CE 191: Civil and Environmental Engineering Systems Analysis

LEC 01 : Linear Programming

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Canonical Optimization Problem

Summary of Entire Course

Minimize a cost (or objective) function:

$$\min_{x} \quad f(x),$$

subject to constraints:

$$\begin{aligned} g_i(x) &\leq 0, \quad i = 1, \cdots, m \\ h_j(x) &= 0, \quad j = 1, \cdots, l. \end{aligned}$$

Objective Function - What is it?

$$\min_{x \in \mathcal{D}} \quad f(x)$$

x | decision variables

$x \in \mathcal{D} \subseteq R^n$ | decision variables live in space $\mathcal{D}$

$f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ | maps decision variables into performance metric

Example objectives:

- Monetary cost [USD]
- Fuel consumption [gal]
- Emissions [grams]
- Power [kW]
- ...

Objective Function Remarks

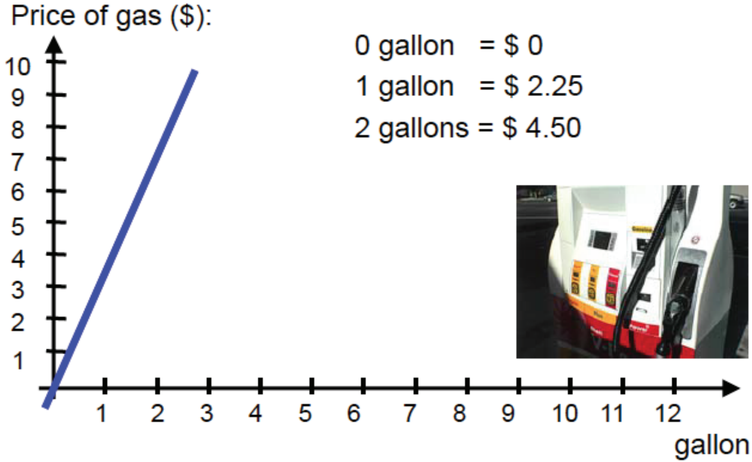
Remark 1 (Objective Function Terminology).
Note “cost”, “reward”, and “objective” function are used interchangeably.

Remark 2 (Maximize a Reward).
Maximizing any “reward” function $f(x)$ is equivalent to minimizing $-f(x)$.

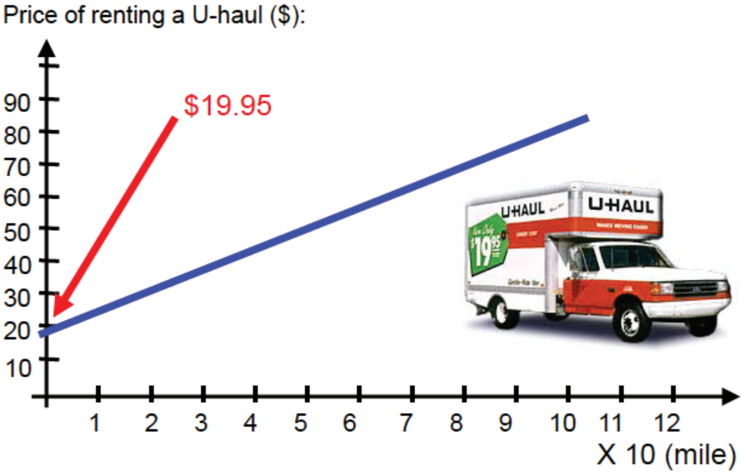
$$x^* = \arg\max_x f(x) = \arg\min_x -f(x)$$

Linear Function

0 gallon = \$ 0
1 gallon = \$ 2.25
2 gallons = \$ 4.50



Affine Function



Linear vs. Affine Cost Functions

Minimizing the affine cost function

$$f(x_1, x_2) = 2x_1 + 3x_2 + 5$$

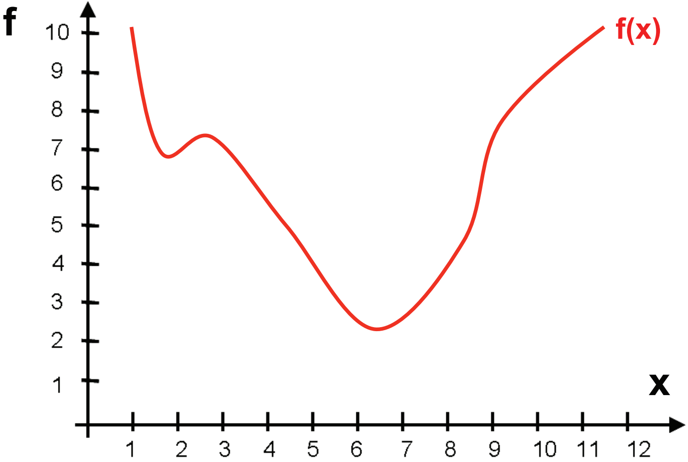
is the same as minimizing the linear cost function

$$f(x_1, x_2) = 2x_1 + 3x_2$$

A general expression for a linear cost function is

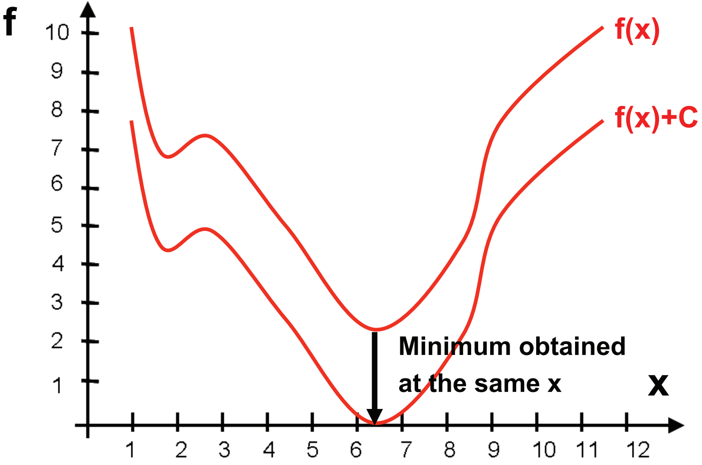
$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \\ &= \sum_{i=1}^{n} a_i x_i \\ &= a^T x \end{aligned}$$

Minimizing a function $f(x)$



Linear vs. Affine | Graphical Explanation

Minimizing a function $f(x)$ or $f(x) + C$ is the same



Example: Building a wall

Cost of a pound of cement (USD per lbs)	a_1
Cost of a foot of steel beam (USD per ft)	a_2
Weight of cement (lbs)	x_1
Length of steel beam (ft)	x_2
Total cost (USD)	$f(x_1, x_2) = a_1 x_1 + a_2 x_2$

Note: All variables have different dimensions

Note: Expressions $a_1 x_1$, $a_2 x_2$, and $f(x_1, x_2)$ have the same units - USD.

Constraints - What are they?

Constraints encode physical restrictions on the decision variables.

Your maximum budget for cement is c_{max}

$$a_1 x_1 \leq c_{\text{max}}$$

Your minimum budget for steel is s_{min}

$$a_2 x_2 \geq s_{\text{min}}$$

You cannot have negative pounds of cement

$$x_1 \geq 0$$

You maximum total budget is f_{max}

$$a_1 x_1 + a_2 x_2 \leq f_{\text{max}}$$

Summary of Program

The optimization program incorporating all the constraints can be formulated as:

$$\begin{aligned} \text{Minimize:} \quad & f(x_1, x_2) = a_1 x_1 + a_2 x_2 \\ \text{Subject to:} \quad & a_1 x_1 \le c_{\text{max}} \\ & a_2 x_2 \ge s_{\text{min}} \\ & x_1 \ge 0 \\ & a_1 x_1 + a_2 x_2 \le f_{\text{max}} \end{aligned}$$

Summary of Program

The optimization program incorporating all the constraints can be formulated as:

$$
\begin{aligned}
\text{Minimize:} \quad & f(x_1, x_2) = a_1 x_1 + a_2 x_2 \\
\text{Subject to:} \quad & a_1 x_1 - c_{\text{max}} \leq 0 \\
& s_{\text{min}} - a_2 x_2 \leq 0 \\
& -x_1 \leq 0 \\
& a_1 x_1 + a_2 x_2 - f_{\text{max}} \leq 0
\end{aligned}
$$

Negative-null form

Summary of Program

The optimization program incorporating all the constraints can be formulated as:

$$
\begin{aligned}
\text{Minimize:} \quad & f(x_1, x_2) = a_1 x_1 + a_2 x_2 \\
\text{Subject to:} \quad & a_1 x_1 \leq c_{\max} \\
& -a_2 x_2 \leq -s_{\min} \\
& -x_1 \leq 0 \\
& a_1 x_1 + a_2 x_2 \leq f_{\max}
\end{aligned}
$$

Standard form

Equality Constraints

Sometimes, physical constraints take the mathematical form of equalities.

Ex: You must spend exactly twice as much for steel as for cement:

$$a_2 x_2 = 2 a_1 x_1$$

Remark 1: This is exactly the same as

$$a_2 x_2 \geq 2 a_1 x_1 \qquad \text{AND} \qquad a_2 x_2 \leq 2 a_1 x_1$$

Equality Constraints $\rightarrow$ Inequality Constraints

$$\begin{aligned}
\text{Minimize:} \quad & f(x_1, x_2) = a_1 x_1 + a_2 x_2 \\
\text{Subject to:} \quad & a_1 x_1 \leq c_{\text{max}} \\
& a_2 x_2 \geq s_{\text{min}} \\
& x_1 \geq 0 \\
& a_1 x_1 + a_2 x_2 \leq f_{\text{max}} \\
& a_2 x_2 \geq 2 a_1 x_1 \\
& a_2 x_2 \leq 2 a_1 x_1
\end{aligned}$$

One can thus assume all constraints are always given in the form of inequalities.

Equality Constraints

Sometimes, physical constraints take the mathematical form of equalities.

Ex: You must spend exactly twice as much for steel as for cement:

$$a_2 x_2 = 2 a_1 x_1$$

Remark 2: By solving the equality constraint, one can reduce the problem size

$$x_2 = \frac{2a_1}{a_2} x_1$$

Equality Constraints → Program Reduction

$$
\begin{aligned}
\text{Minimize:} \quad & f(x_1) = 3a_1x_1 \\
\text{Subject to:} \quad & a_1x_1 \le c_{\max} \\
& 2a_1x_1 \ge s_{\min} \\
& x_1 \ge 0 \\
& 3a_1x_1 \le f_{\max}
\end{aligned}
$$

Equality Constraints → Program Reduction

$$\begin{aligned} \text{Minimize:} \quad & f(x_1) = 3a_1x_1 \\ \text{Subject to:} \quad & a_1x_1 \leq c_{\max} \\ & 2a_1x_1 \geq s_{\min} \\ & x_1 \geq 0 \\ & 3a_1x_1 \leq f_{\max} \end{aligned}$$

Solution:

$$f(x_1^*) = \min_{x_1} \; f(x_1) = \frac{3}{2}s_{\min}$$

$$x_1^* = \arg\min_{x_1} \; f(x_1) = \frac{s_{\min}}{2a_1}$$

Note: We use the asterisk notation to denote an optimum.

Equality Constraints $\rightarrow$ Program Reduction

$$
\begin{aligned}
\text{Minimize:} \quad & f(x_1) = 3a_1x_1 \\
\text{Subject to:} \quad & a_1x_1 \leq c_{\max} \\
& 2a_1x_1 \geq s_{\min} \\
& x_1 \geq 0 \\
& 3a_1x_1 \leq f_{\max}
\end{aligned}
$$

Solution:

$$f(x_1^*) = \min_{x_1} \; f(x_1) = \frac{3}{2}s_{\min}$$

$$x_1^* = \arg\min_{x_1} \; f(x_1) = \frac{s_{\min}}{2a_1}$$

Note: We use the asterisk notation to denote an optimum.
When an inequality constraint is true with EQUALITY at the optimal solution, we say it is ACTIVE.

General Form of LP

$$
\begin{aligned}
\text{Minimize:} \quad & c_1 x_1 + c_2 x_2 + \ldots + c_N x_N \\
\text{subject to:} \quad & a_{1,1} x_1 + a_{1,2} x_2 + \ldots + a_{1,N} x_N \leq b_1 \\
& a_{2,1} x_1 + a_{2,2} x_2 + \ldots + a_{2,N} x_N \leq b_2 \\
& \vdots \\
& a_{M,1} x_1 + a_{M,2} x_2 + \ldots + a_{M,N} x_N \leq b_M
\end{aligned}
$$

General Form of LP

"Sigma notation":

$$
\begin{aligned}
\text{Minimize:} \quad & \sum_{k=1}^{N} c_k x_k \\
\text{subject to:} \quad & \sum_{k=1}^{N} a_{1,k} x_k \leq b_1 \\
& \sum_{k=1}^{N} a_{2,k} x_k \leq b_2 \\
& \vdots \\
& \sum_{k=1}^{N} a_{M,k} x_k \leq b_M
\end{aligned}
$$

General Form of LP

"Matrix notation":

$$\begin{aligned} \text{Minimize:} \quad & c^T x \\ \text{subject to:} \quad & Ax \leq b \end{aligned}$$

where

$$\begin{aligned} x &= [x_1, x_2, \ldots, x_N]^T \\ c &= [c_1, c_2, \ldots, c_N]^T \\ [A]_{i,j} &= a_{i,j}, \quad A \in \mathbb{R}^{M \times N} \\ b &= [b_1, b_2, \ldots, b_M]^T \end{aligned}$$

Additional Reading

Revelle: Chapter 2 - Models in CEE Systems

- Overview of optimization problems

Revelle: Chapter 4 - Simplex Algorithm for Linear Programs

- Overview of LP
- Dantzig's Simplex algorithm

Boyd: Section 1.2 - Least Squares & Linear Programming

- Brief overview