CE 191: Civil and Environmental Engineering Systems Analysis

LEC 02 : LP Examples

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Example 1: Transportation Problem

Paul's farm produces 4 tons of apples per day	$s_p = 4$
Ron's farm produces 2 tons of apples per day	$s_r = 2$
Max's factory needs 1 ton of apples per day	$d_m = 1$
Bob's factory needs 5 tons of apples per day	$d_b = 5$

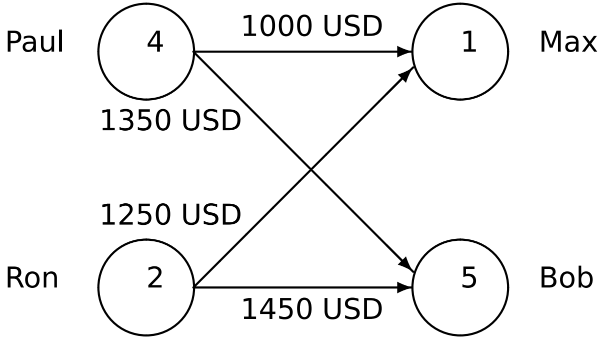
George owns both farms and factories. He is paying the cost of shipping all the apples from the farms to the factories.

The shipping costs for George are:

Paul $\rightarrow$ Max: 1000 USD per ton	$c_{pm} = 1000$	x_{pm}
Ron $\rightarrow$ Max: 1250 USD per ton	$c_{rm} = 1350$	x_{rm}
Paul $\rightarrow$ Bob: 1350 USD per ton	$c_{pb} = 1250$	x_{pb}
Ron $\rightarrow$ Bob: 1450 USD per ton	$c_{rb} = 1450$	x_{rb}

What is the best way to ship the apples?

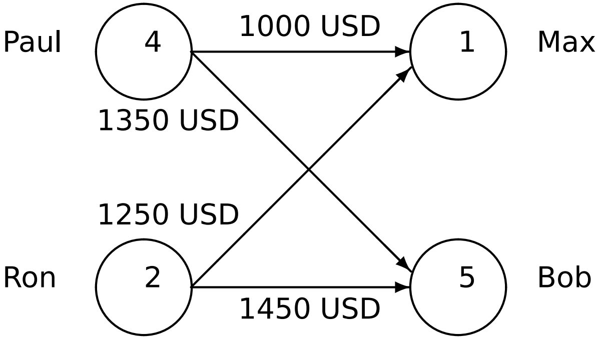
Ex 1: Transportation Problem - Network Graph



Ex 1: Transportation Problem - LP Formulation (I)

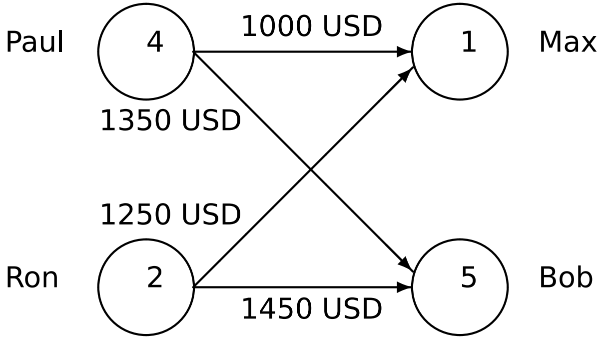
$$\text{min:} \quad 1000x_{pm} + 1350x_{pb} + 1250x_{rm} + 1450x_{rb}$$

$$\begin{aligned} \text{s. to} \qquad x_{pm} + x_{rm} &= 1 \\ x_{pb} + x_{rb} &= 5 \\ x_{pm} + x_{pb} &= 4 \\ x_{rm} + x_{rb} &= 2 \\ x_{pm} \geq 0, x_{pb} \geq 0, x_{rm} \geq 0, x_{rb} &\geq 0 \end{aligned}$$



Ex 1: Transportation Problem - LP Formulation (II)

$$
\begin{aligned}
\text{min:} \quad & c_{pm}x_{pm} + c_{pb}x_{pb} + c_{rm}x_{rm} + c_{rb}x_{rb} \\
\text{s. to} \quad & x_{pm} + x_{rm} = d_m \\
& x_{pb} + x_{rb} = d_b \\
& x_{pm} + x_{pb} = s_p \\
& x_{rm} + x_{rb} = s_r \\
& x_{pm} \geq 0, x_{pb} \geq 0, x_{rm} \geq 0, x_{rb} \geq 0
\end{aligned}
$$



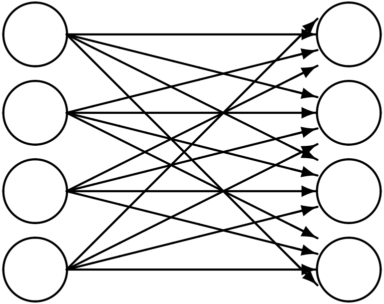
Ex 1: General LP Formulation

$$\min: \quad \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} x_{ij}$$

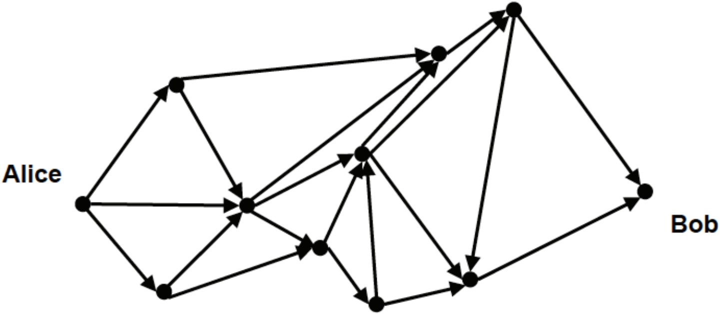
$$\text{s. to} \quad \sum_{i=1}^{M} x_{ij} = d_j, \qquad j = 1, \cdots, N$$

$$\sum_{j=1}^{N} x_{ij} = s_i, \qquad i = 1, \cdots, M$$

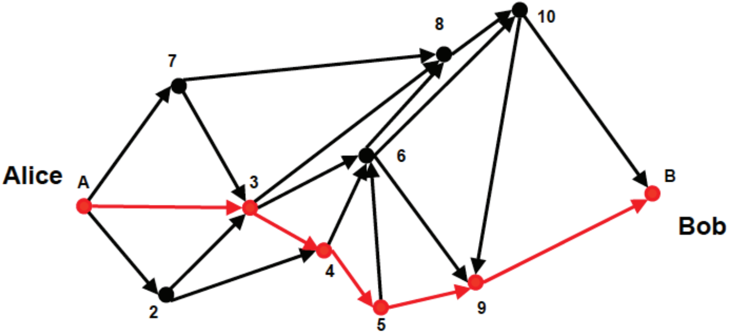
$$x_{ij} \geq 0, \qquad \forall i, j$$



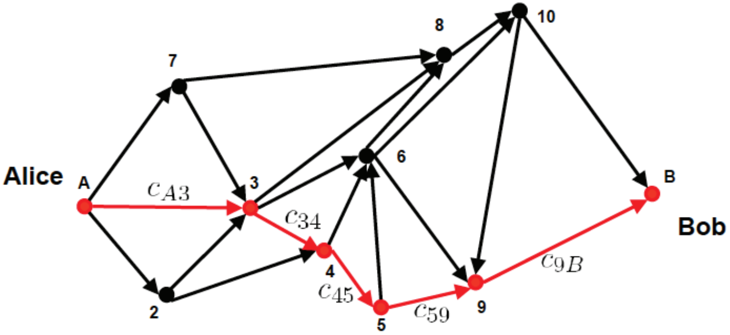
Example 2: Shortest Path



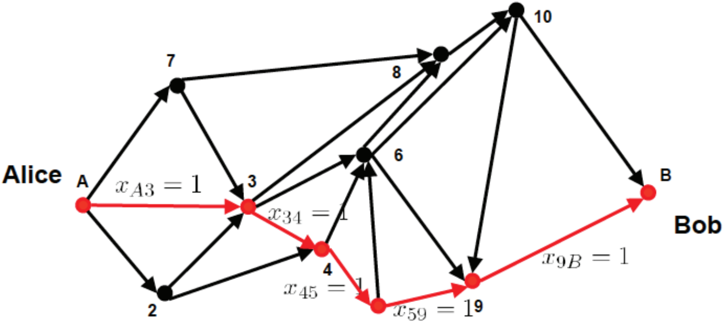
Example 2: Shortest Path



Example 2: Shortest Path



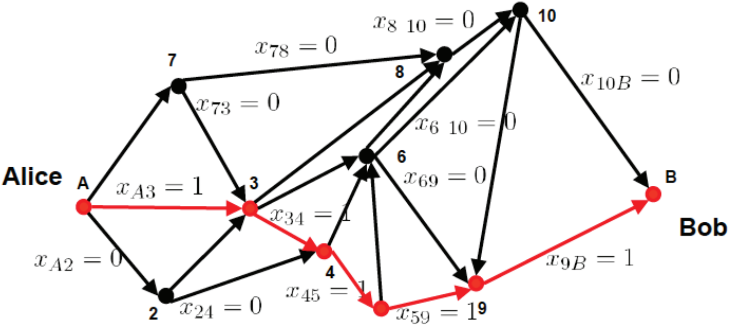
Example 2: Shortest Path



Define

$$x_{ij} = 1 \quad \text{For every } (i,j) \text{ on the shortest path}$$
$$x_{ij} = 0 \quad \text{For every } (i,j) \text{ not on the shortest path}$$

Example 2: Shortest Path



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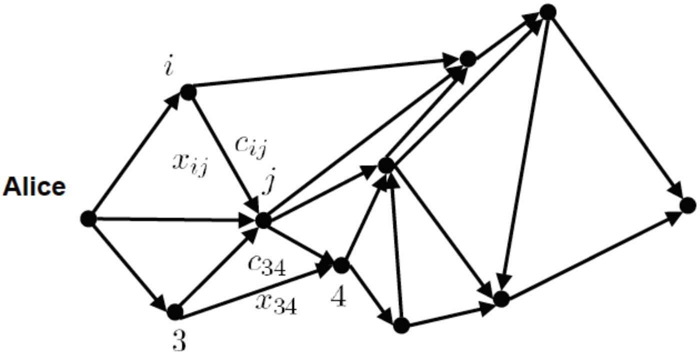
Example 2: Shortest Path

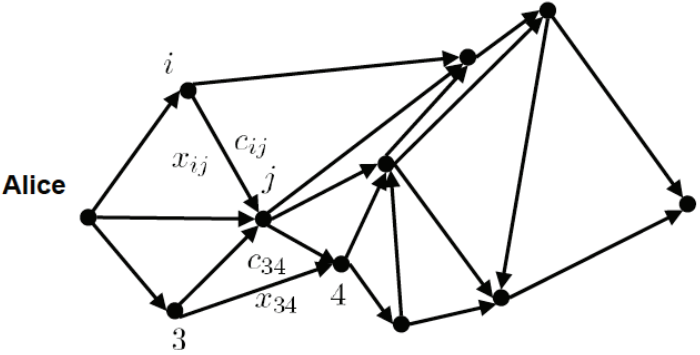


Define

$x_{ij} = 1$ For every (i, j) on the shortest path
$x_{ij} = 0$ For every (i, j) not on the shortest path

Example 2: Shortest Path

Define a graph (road network)

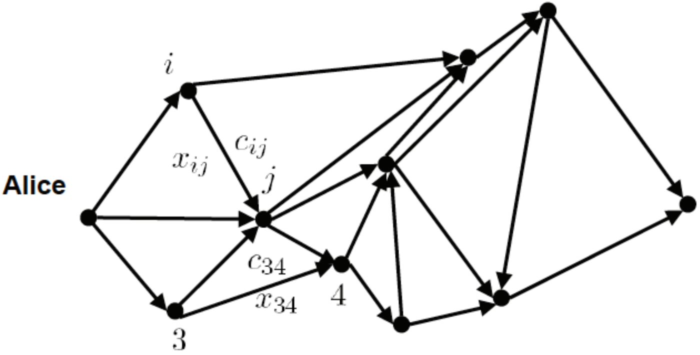


Denote c_{ij} as the cost to go from i to j (e.g. fuel burned)

For example c_{34} is the cost to go from node 3 to node 4

Example 2: Shortest Path

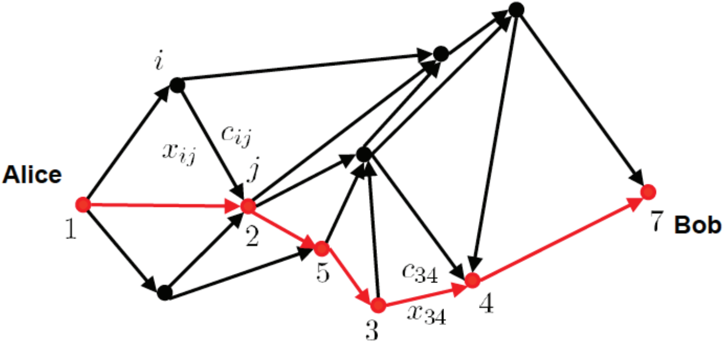
Define a graph (road network)



Take $x_{ij} = 1$ if Alice decides to go through link (i, j), zero otherwise

For example $x_{34} = 1$ if Alice decides to use route (3,4)

Example 2: Shortest Path

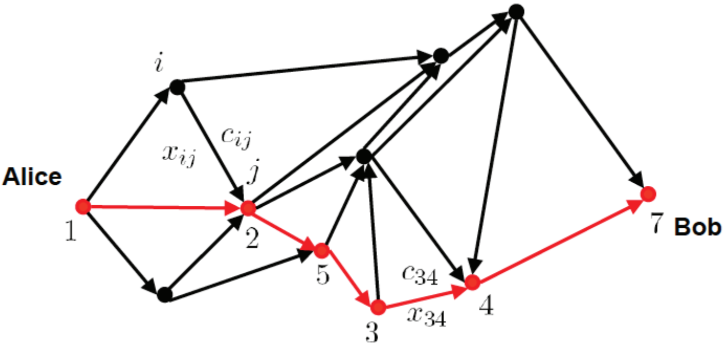


$x_{12} = x_{25} = x_{53} = x_{34} = x_{47} = 1$

All other $x_{ij} = 0$

Total length of this path: $c_{12} + c_{25} + c_{53} + c_{34} + c_{47}$

Example 2: Shortest Path



Total length:

$$\sum_{(i,j)\text{ chosen on path}} c_{ij} = \sum_{(i,j)\text{ chosen on path}} c_{ij}x_{ij} = \sum_{\text{all }(i,j)} c_{ij}x_{ij}$$

Example 2: Shortest Path

Minimize: $$J = \sum_{j \in \mathcal{D}_A} c_{Aj} x_{Aj} + \sum_{i=1}^{10} \sum_{j \in \mathcal{D}_i} c_{ij} x_{ij} + \sum_{j \in \mathcal{A}_B} c_{iB} x_{iB}$$

$\mathcal{D}_A, \mathcal{D}_i$: Subset of nodes that descend from nodes A and i, respectively.

Ex: $\mathcal{D}_A = \{2, 3, 7\}, \quad \mathcal{D}_5 = \{6, 9\}$.

$\mathcal{A}_B$: Subset of nodes that ascend from node B.

Ex: $\mathcal{A}_B = \{9, 10\}$.

Example 2: Shortest Path

Minimize:
$$J = \sum_{j \in \mathcal{D}_A} c_{Aj} x_{Aj} + \sum_{i=1}^{10} \sum_{j \in \mathcal{D}_i} c_{ij} x_{ij} + \sum_{j \in \mathcal{A}_B} c_{iB} x_{iB}$$

subject to:
$$\sum_{i \in \mathcal{A}_j} x_{ij} = \sum_{k \in \mathcal{D}_j} x_{jk}, \quad j = 1, \cdots, 10, \quad \text{[leg stiching]}$$

$\mathcal{D}_A, \mathcal{D}_i$: Subset of nodes that descend from nodes A and i, respectively.

Ex: $\mathcal{D}_A = \{2, 3, 7\}, \quad \mathcal{D}_5 = \{6, 9\}$.

$\mathcal{A}_B$: Subset of nodes that ascend from node B.

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subject to:
$$\sum_{i \in \mathcal{A}_j} x_{ij} = \sum_{k \in \mathcal{D}_j} x_{jk}, \quad j = 1, \cdots, 10, \quad \text{[leg stiching]}$$
$$\sum_{j \in N_A} x_{Aj} = 1, \quad \text{[origin]}$$

$\mathcal{D}_A, \mathcal{D}_i$: Subset of nodes that descend from nodes A and i, respectively.
Ex: $\mathcal{D}_A = \{2, 3, 7\}, \quad \mathcal{D}_5 = \{6, 9\}$.
$\mathcal{A}_B$: Subset of nodes that ascend from node B.
Ex: $\mathcal{A}_B = \{9, 10\}$.

Example 2: Shortest Path

$$\text{Minimize:} \qquad J = \sum_{j \in \mathcal{D}_A} c_{Aj} x_{Aj} + \sum_{i=1}^{10} \sum_{j \in \mathcal{D}_i} c_{ij} x_{ij} + \sum_{j \in \mathcal{A}_B} c_{iB} x_{iB}$$

$$\begin{aligned} \text{subject to:} \qquad \sum_{i \in \mathcal{A}_j} x_{ij} &= \sum_{k \in \mathcal{D}_j} x_{jk}, \quad j = 1, \cdots, 10, \quad \text{[leg stiching]} \\ \sum_{j \in N_A} x_{Aj} &= 1, \quad \text{[origin]} \\ \sum_{j \in N_B} x_{jB} &= 1, \quad \text{[destination]} \end{aligned}$$

$\mathcal{D}_A, \mathcal{D}_i$: Subset of nodes that descend from nodes *A* and *i*, respectively.

Ex: $\mathcal{D}_A = \{2, 3, 7\}, \quad \mathcal{D}_5 = \{6, 9\}$.

$\mathcal{A}_B$: Subset of nodes that ascend from node *B*.

Ex: $\mathcal{A}_B = \{9, 10\}$.

Example 2: Shortest Path

$$\text{Minimize:} \qquad J = \sum_{j \in \mathcal{D}_A} c_{Aj} x_{Aj} + \sum_{i=1}^{10} \sum_{j \in \mathcal{D}_i} c_{ij} x_{ij} + \sum_{j \in \mathcal{A}_B} c_{iB} x_{iB}$$

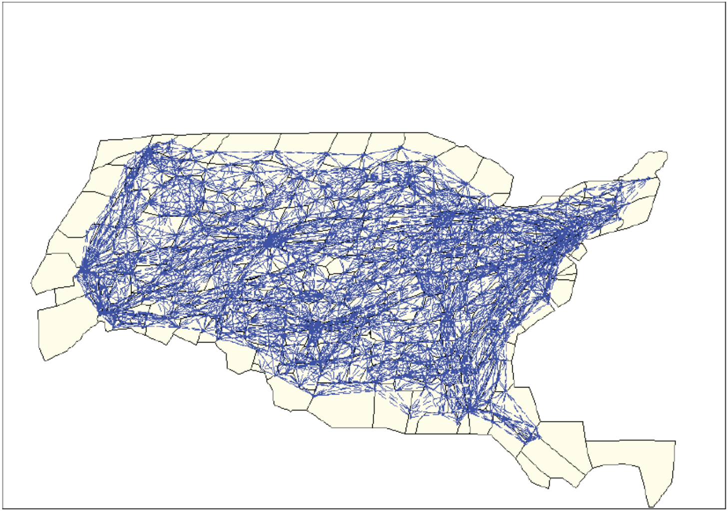
$$\begin{aligned} \text{subject to:} \qquad \sum_{i \in \mathcal{A}_j} x_{ij} &= \sum_{k \in \mathcal{D}_j} x_{jk}, \quad j = 1, \cdots, 10, \quad \text{[leg stiching]} \\ \sum_{j \in N_A} x_{Aj} &= 1, \quad \text{[origin]} \\ \sum_{j \in N_B} x_{jB} &= 1, \quad \text{[destination]} \end{aligned}$$

$$x_{ij} \geq 0, \ \forall\, i, j \in \{1, \cdots, 10\}, \qquad x_{Aj} \geq 0, \ \forall\, j \in \mathcal{D}_A, \qquad x_{iB} \geq 0, \ \forall\, i \in \mathcal{A}_B.$$

$\mathcal{D}_A, \mathcal{D}_i$: Subset of nodes that descend from nodes A and i, respectively.
Ex: $\mathcal{D}_A = \{2, 3, 7\}, \quad \mathcal{D}_5 = \{6, 9\}$.

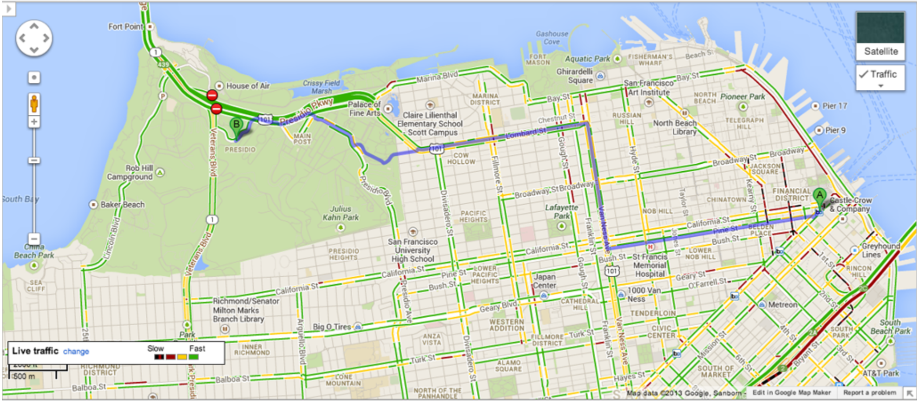
$\mathcal{A}_B$: Subset of nodes that ascend from node B.
Ex: $\mathcal{A}_B = \{9, 10\}$.

Example: a small network (air traffic control)

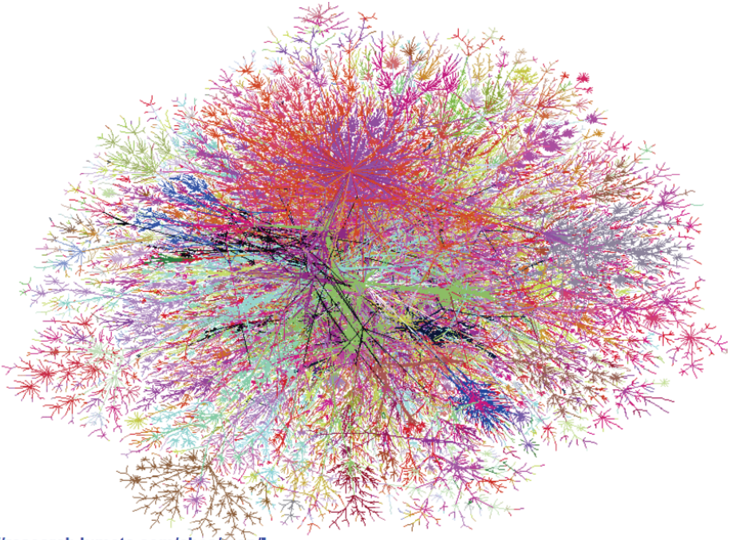


[Robelin, Sun, Bayen, tech. rep., 2005]

Example: a medium network (US roads)



Example: a large network (the internet)



[http://research.lumeta.com/ches/map/]

Additional Reading

Revelle

- Chapter 6.D - The transportation problem
- Chapter 6.B - The shortest path problem

Related LP problems of interest in Revelle:

- Chapter 6.E - The transshipment problem
- Chapter 6.F - The maximum flow problem
- Chapter 6.G - The traveling salesman problem