CE 191: Civil and Environmental Engineering Systems Analysis

LEC 04 : Introduction to Quadratic Programs (QP)

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General Form of Quadratic Program (QP)

$$\text{Minimize:} \qquad \frac{1}{2}x^T Q x + R^T x + S$$

$$\text{subject to:} \qquad Ax \leq b$$

$$A_{eq}x = b_{eq}$$

Design Vars: $x \in \mathbb{R}^n$

$Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^n$, $S \in \mathbb{R}$,
$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $A_{eq} \in \mathbb{R}^{l \times n}$, $b_{eq} \in \mathbb{R}^l$,

n = # of design variables,
m = # of inequality constraints,
l = # of equality constraints.

Remarks on QP Format

(1) Can drop "S" term without loss of generality

$$\text{Minimize:} \quad \frac{1}{2}x^T Q x + R^T x$$

$$\text{subject to:} \quad Ax \leq b$$
$$A_{eq}x = b_{eq}$$

(2) Quadratically Constrained Quadratic Program (QCQP).

$$\text{Minimize:} \quad \frac{1}{2}x^T Q x + R^T x$$

$$\text{subject to:} \quad \frac{1}{2}x^T U x + V^T x + W \leq 0$$
$$A_{eq}x = b_{eq}$$

Solvers exist. Not discussed in CE 191.

Overdetermined Systems

Consider a “skinny” matrix A, and the following equation to solve

$$y = Ax$$

x : unknown
y : measured data
$A \in \mathbb{R}^{m \times n}$: known matrix, where $m > n$

Least Squares Solution

This set of equations is called "overdetermined", since there are more equations than unknowns.

For most y, it is not possible to find a unique solution for x.

One possible approach is to look for an approximate solution.
For this, one can define the residual error, defined by

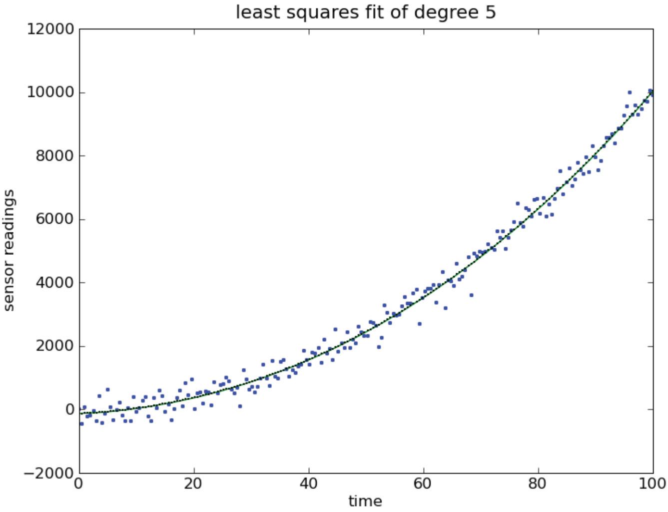
$$r = Ax - y$$

Consider the solution x^* that minimizes $\|r\|$.

This solution is called the "least squares" solution to the problem.

Graphical Example

The curve-fit Ax^* is closest to the measured data y, in a least squares sense.



Least Squares Solution

The least squares solution is given by the famous formula

$$x^* = (A^T A)^{-1} A^T y$$

where some assumptions make the matrix $(A^T A)$ invertible.

Remark: The matrix $(A^T A)^{-1} A^T$ is sometimes called the *pseudoinverse*.

Derivation

Assume:

1. A is full rank
2. A is skinny, i.e. $A \in \mathbb{R}^{m \times n}$, $m > n$.

To find x^*, we minimize the norm of the residual squared,

$$\|r\|^2 = x^T A^T A x - 2y^T A x + y^T y$$

Set the gradient w.r.t. x equal to zero,

$$\frac{\partial}{\partial x}\|r\|^2 = 2A^T A x - 2A^T y = 0$$

which yields the *normal equations*

$$A^T A x = A^T y$$

The assumptions imply $A^T A$ is invertible, so we have

$$\boxed{x^* = (A^T A)^{-1} A^T y}$$

What does it mean to do least squares?

Reflection on minimizing the residual squared

$$\begin{aligned} \|Ax - y\|^2 &= (Ax - y)^T(Ax - y) \\ &= x^T A^T A x - 2y^T A x + y^T y \\ &= x^T Q x^T + Rx + S \end{aligned}$$

Least squares can then be viewed as minimizing a quadratic cost function:

$$\min \quad x^T Q x + Rx + S$$

This is a quadratic program (QP)!

Remark 1: If the problem is unconstrained, then the solution takes the closed-form solution (using the psuedoinverse) as derived above.

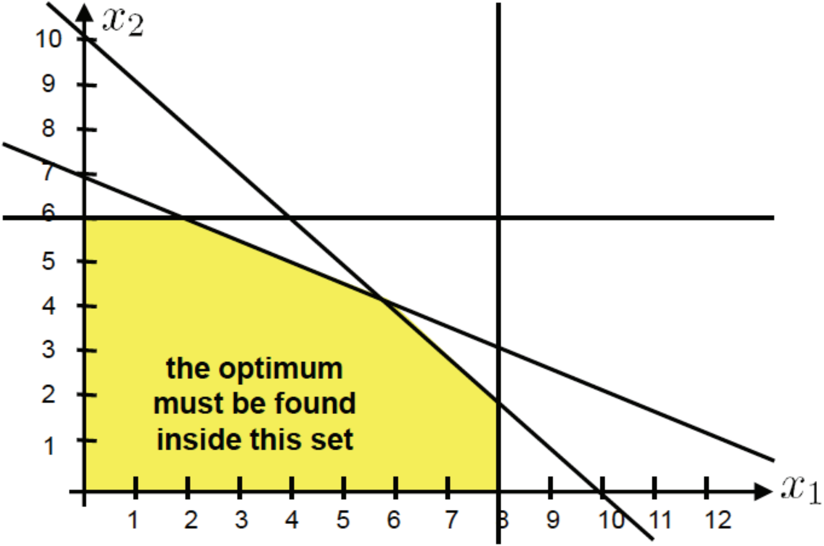
Remark 2: The same minimizer x^* solves min $\left(x^T Q x + Rx + S\right)$ and min $\left(x^T Q x + Rx\right)$.

Graphical QP

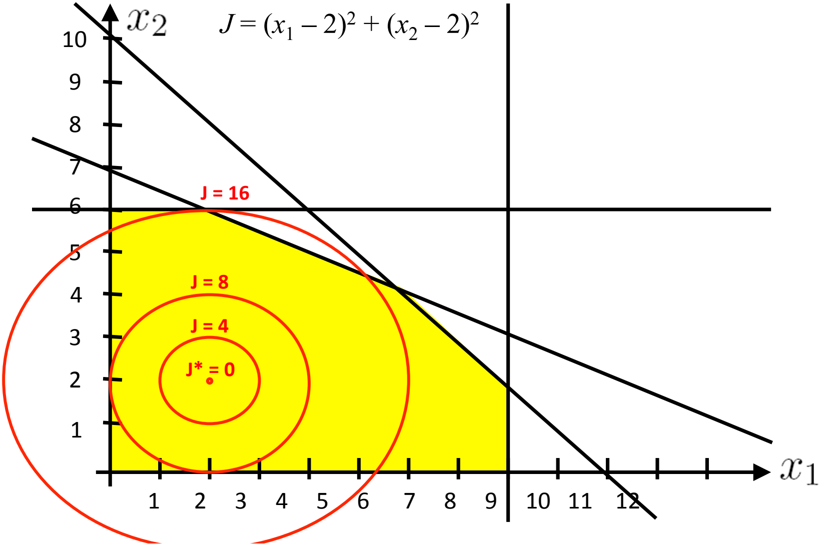
Consider the familiar example with the same linear constraints, but now with a quadratic cost function

$$\begin{aligned} \min \qquad & J = (x_1 - 2)^2 + (x_2 - 2)^2 \\ \text{s. to} \qquad & 2x_1 + 4x_2 \leq 28 \\ & 5x_1 + 5x_2 \leq 50 \\ & x_1 \leq 8 \\ & x_2 \leq 6 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

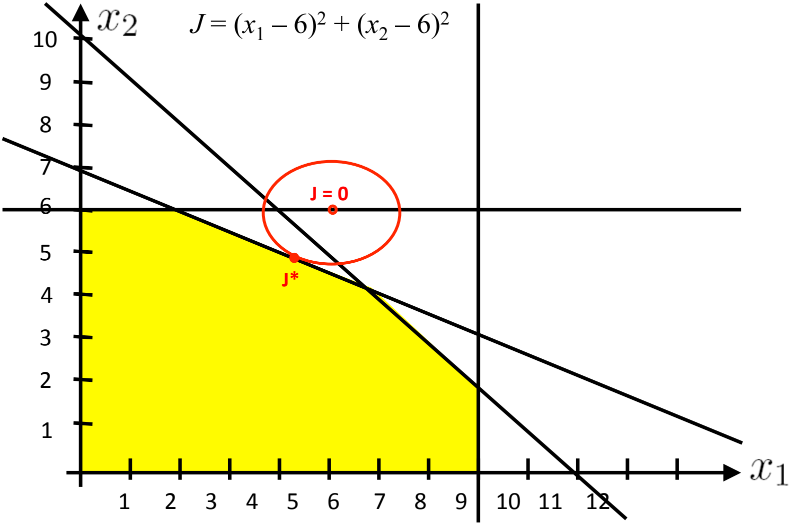
The feasible set is the same as before



Draw isolines - Solution is an *interior* point



Ex2 : Solution can be *boundary* point



Additional Reading

Simplex Algorithm

- Revelle Chapter 4 - The Simplex Algorithm for Solving Linear Programs

Quadratic cost functions

- Papalambros & Wilde Section 4.2 - Local Approximations
- Papalambros & Wilde Section 4.3 - Optimality Conditions