

CE 191: Civil and Environmental Engineering Systems Analysis

LEC 08 : Branch & Bound

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Civil & Environmental Engineering
University of California, Berkeley

Fall 2014

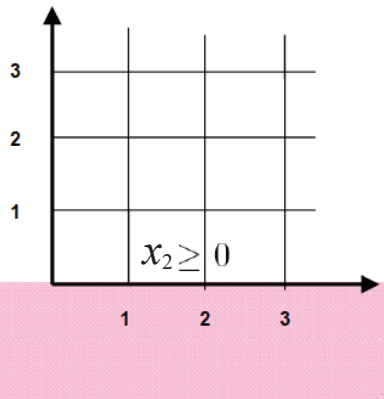


Problem Statement

Solve integer programming problem with no special structure:

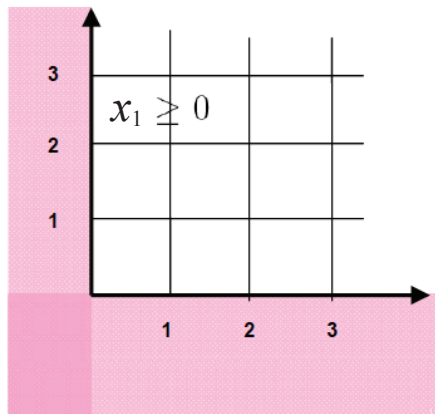
$$\begin{array}{ll} \min & c^T x \\ \text{s. to} & Ax \leq b \\ & x \in \mathbb{Z} \end{array}$$

Reminder: Feasible set of a LP



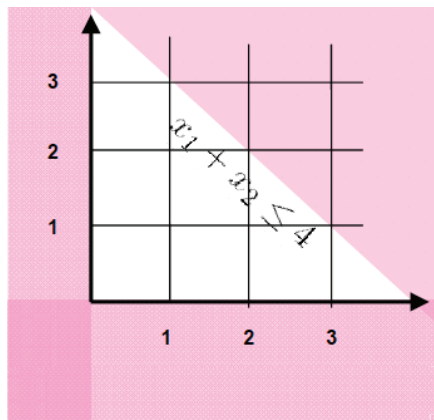
$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Reminder: Feasible set of a LP



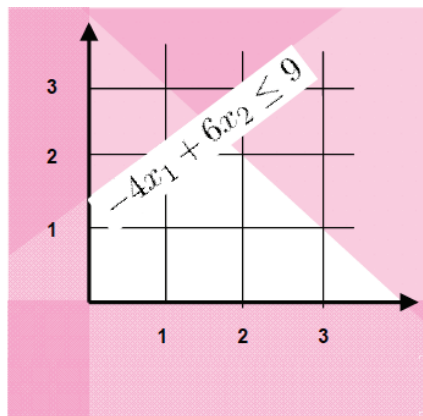
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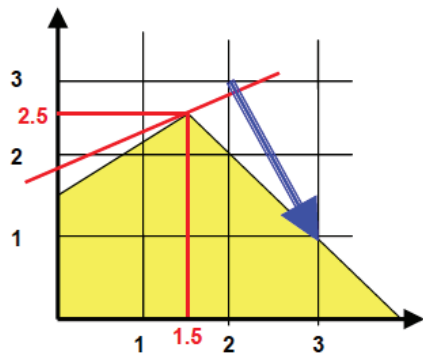
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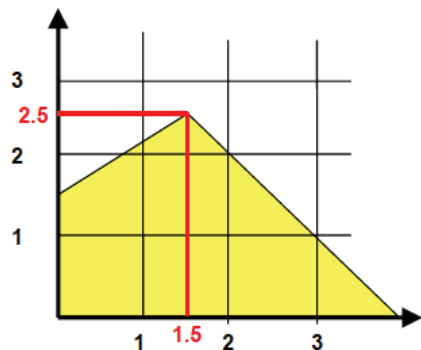
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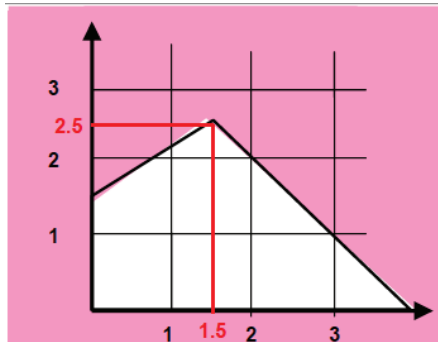
Branch and Bound algorithm



Solve LP: $x^* = (1.5, 2.5)$, $f^* = -3.5$

$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Problem P1: add constraint $x_2 \geq 3$

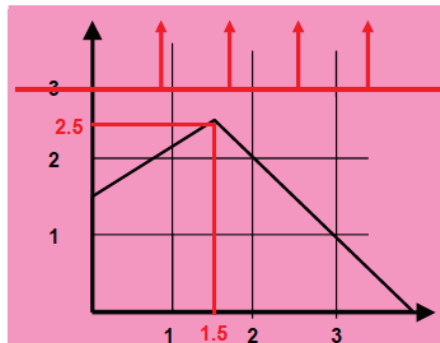


Solve P0: $x^* = (1.5, 2.5)$, $f^* = -3.5$

P1 : Add constraint $x_2 \geq 3$

$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

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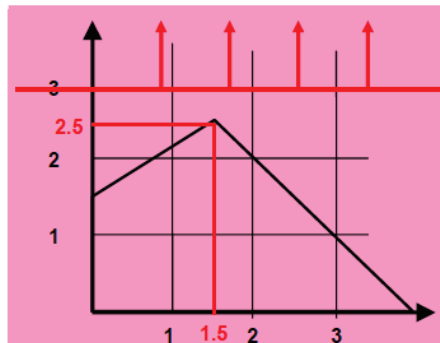


Solve P0: $x^* = (1.5, 2.5)$, $f^* = -3.5$

P1 : Add constraint $x_2 \geq 3$
P1 infeasible

$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Problem P1: discard

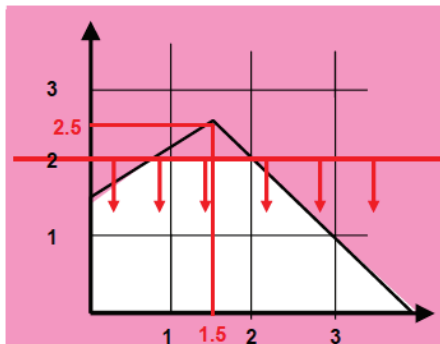


Solve P0: $x^* = (1.5, 2.5)$, $f^* = -3.5$

P1 : Add constraint $x_2 \geq 3$
P1 infeasible \rightarrow DISCARD

$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Problem P2: add constraint $x_2 \leq 2$



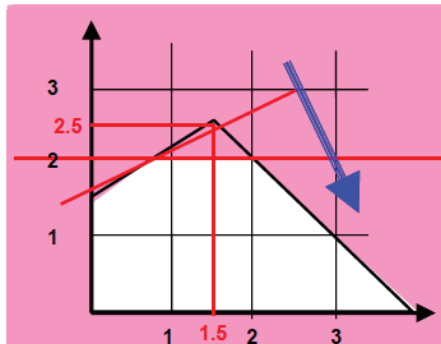
Solve P0: $x^* = (1.5, 2.5)$, $f^* = -3.5$

P1 : Add constraint $x_2 \geq 3$
P1 infeasible \rightarrow DISCARD

P2 : P0 + constraint $x_2 \leq 2$

$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Problem P2: add constraint $x_2 \leq 2$



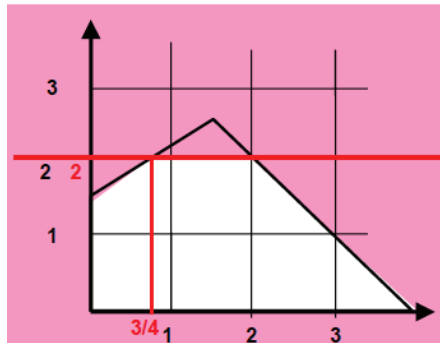
Solve P0: $x^* = (1.5, 2.5)$, $f^* = -3.5$

P1 : Add constraint $x_2 \geq 3$
P1 infeasible \rightarrow DISCARD

P2 : P0 + constraint $x_2 \leq 2$
 $x^* = (0.75, 2)$, $f^* = -3.25$

$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Problem P2: add constraint $x_2 \leq 2$



Solve P0: $x^* = (1.5, 2.5), f^* = -3.5$

P1 : Add constraint $x_2 \geq 3$

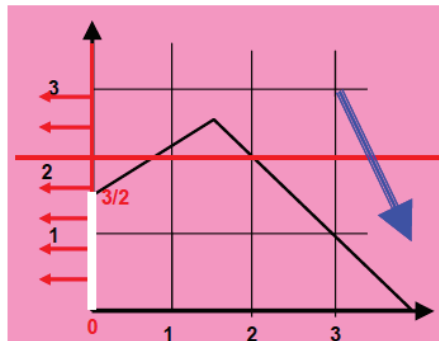
P1 infeasible \rightarrow DISCARD

P2 : P0 + constraint $x_2 \leq 2$

$x^* = (0.75, 2), f^* = -3.25$

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Problem P3: add constraint $x_1 \leq 0$



Solve P0: $x^* = (1.5, 2.5)$, $f^* = -3.5$

P1 : Add constraint $x_2 \geq 3$

P1 infeasible \rightarrow DISCARD

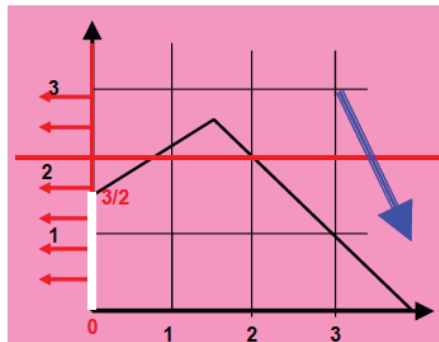
P2 : P0 + constraint $x_2 \leq 2$

$x^* = (0.75, 2)$, $f^* = -3.25$

P3 : P2 + constraint $x_1 \leq 0$

$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Problem P3: add constraint $x_1 \leq 0$



Solve P0: $x^* = (1.5, 2.5), f^* = -3.5$

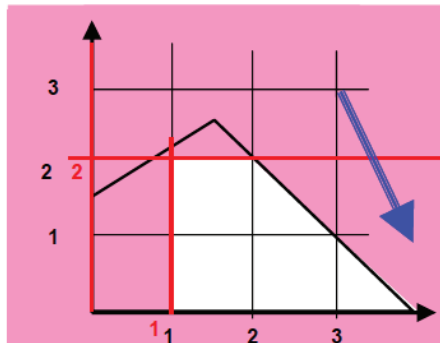
P1 : Add constraint $x_2 \geq 3$
P1 infeasible \rightarrow DISCARD

P2 : P0 + constraint $x_2 \leq 2$
 $x^* = (0.75, 2), f^* = -3.25$

P3 : P2 + constraint $x_1 \leq 0$
 $x^* = (0, 1.5), f^* = -3$

$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Problem P4: add constraint $x_1 \geq 1$



Solve P0: $x^* = (1.5, 2.5)$, $f^* = -3.5$

P1 : Add constraint $x_2 \geq 3$
P1 infeasible \rightarrow DISCARD

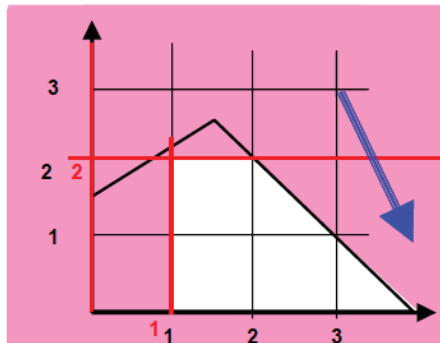
P2 : P0 + constraint $x_2 \leq 2$
 $x^* = (0.75, 2)$, $f^* = -3.25$

P3 : P2 + constraint $x_1 \leq 0$
 $x^* = (0, 1.5)$, $f^* = -3$

P4 : P2 + constraint $x_1 \geq 1$

$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Problem P4: add constraint $x_1 \geq 1$



Solve P0: $x^* = (1.5, 2.5), f^* = -3.5$

P1 : Add constraint $x_2 \geq 3$
P1 infeasible \rightarrow DISCARD

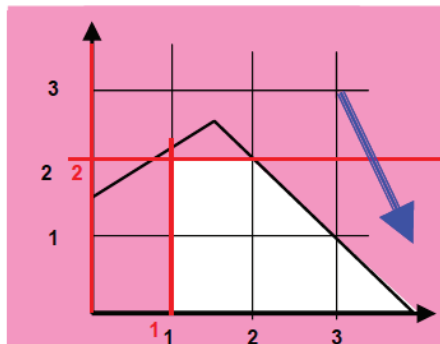
P2 : P0 + constraint $x_2 \leq 2$
 $x^* = (0.75, 2), f^* = -3.25$

P3 : P2 + constraint $x_1 \leq 0$
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P4 : P2 + constraint $x_1 \geq 1$
 $x^* = (1, 2), f^* = -3$

$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Optimum Found



$$\begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s. to} & -4x_1 + 6x_2 \leq 9 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Solve P0: $x^* = (1.5, 2.5), f^* = -3.5$

P1 : Add constraint $x_2 \geq 3$
P1 infeasible \rightarrow DISCARD

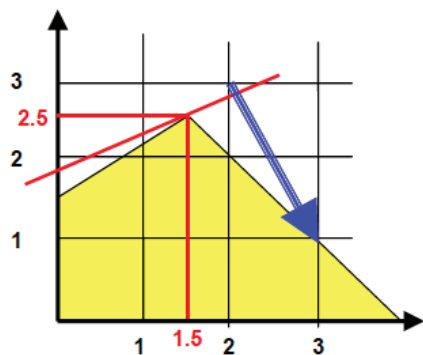
P2 : P0 + constraint $x_2 \leq 2$
 $x^* = (0.75, 2), f^* = -3.25$

P3 : P2 + constraint $x_1 \leq 0$
 $x^* = (0, 1.5), f^* = -3$

P4 : P2 + constraint $x_1 \geq 1$
 $x^* = (1, 2), f^* = -3$

Optimum is $x^* = (1, 2), f^* = -3$
TERMINATE

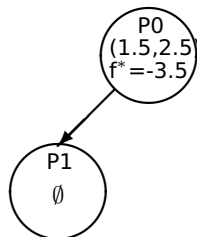
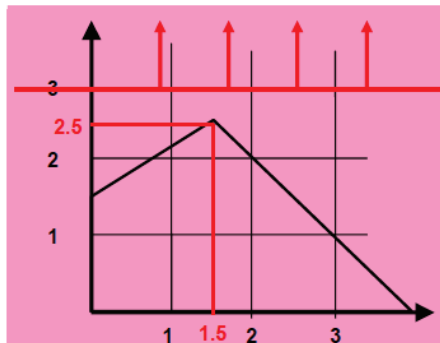
Branch and bound: summary



$$\begin{array}{l} P0 \\ (1.5, 2.5) \\ f^* = -3.5 \end{array}$$

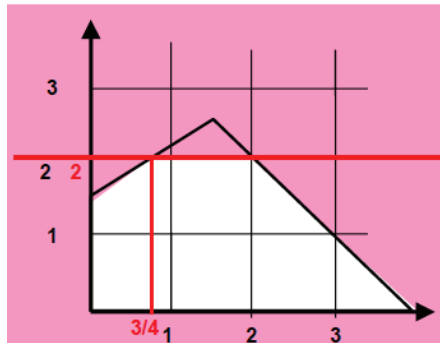
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Branch and bound: summary

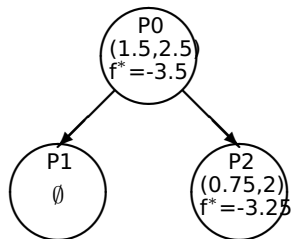


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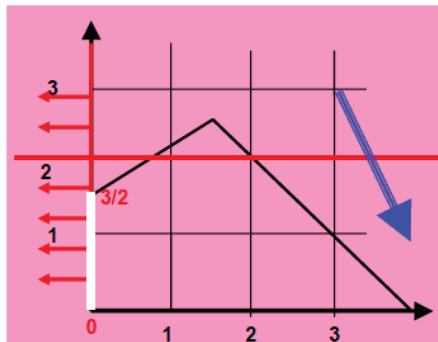
Branch and bound: summary



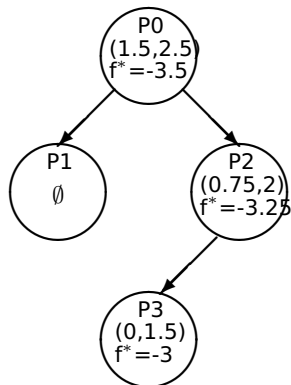
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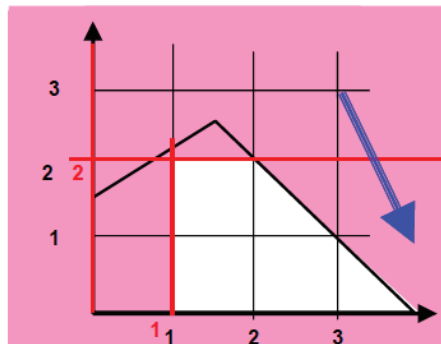
Branch and bound: summary



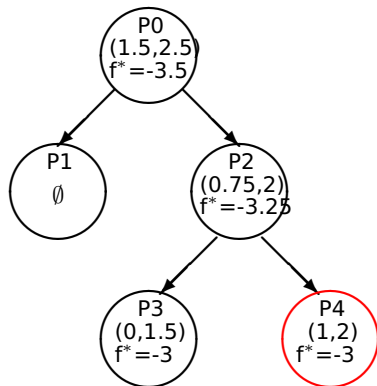
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Branch and bound: summary



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A generic branch and bound algorithm

- 1 Get bound on optimum by solving relaxed LP
- 2 For variable with fractional sol'n, add constraints bounding away
- 3 Solve subproblem. If infeasible, then stop. If feasible, compute optimum.
- 4 Go to step 2. Repeat until optimum has integer solution.

Revelle Section 7.C - Solving IP problems that do not have special structure