CE 191: Civil and Environmental Engineering Systems Analysis

LEC 10 : Intro to Nonlinear Programming

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Intro to Nonlinear Programming

Abstract Optimization Problem

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s. to} \quad & g(x) \leq 0 \\
& h(x) = 0
\end{aligned}
$$

Questions / Issues

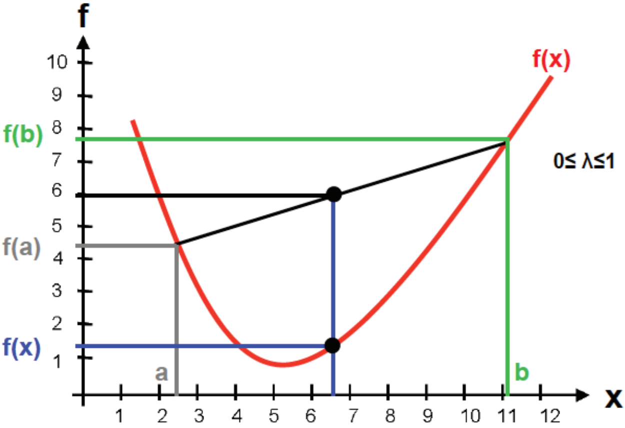
1. What, exactly, is the definition of a minimum
2. Does a solution even exist?
3. Is it unique?
4. What are the necessary & sufficient conditions to be a solution?
5. How do we solve?

Convex Functions

Let $D = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.

Def'n (Convex function) : The function $f(x)$ is convex on D if and only if

$$f(x) = f\left(\lambda a + (1-\lambda)b\right) \leq \lambda f(a) + (1-\lambda) f(b)$$

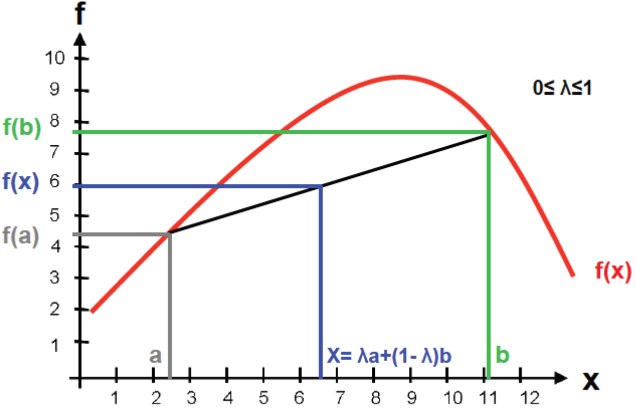


Concave Functions

Let $D = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.

Def'n (Convex function) : The function $f(x)$ is <u>concave</u> on D if and only if

$$f(x) = f(\lambda a + (1 - \lambda)b) \geq \lambda f(a) + (1 - \lambda) f(b)$$



Properties of Convex/Concave Functions

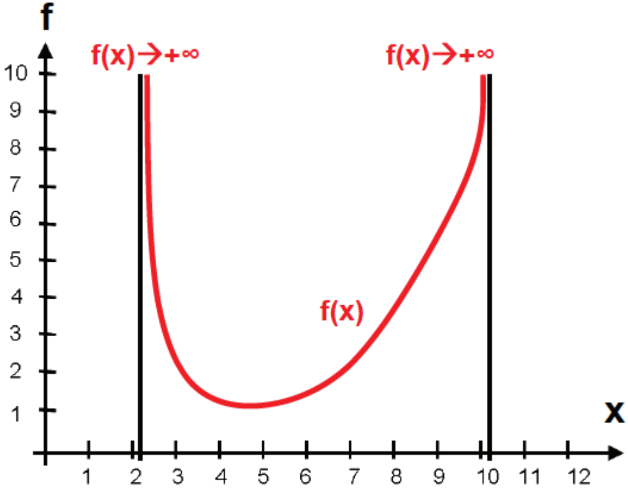
True Properties

- If f is convex, then $-f$ is concave
- If f is concave, then $-f$ is convex
- $f(x)$ is a convex function on $D \Leftrightarrow f''(x)$ is positive semi-definite $\forall x \in D$.
- $f(x)$ is a concave function on $D \Leftrightarrow f''(x)$ is negative semi-definite $\forall x \in D$.

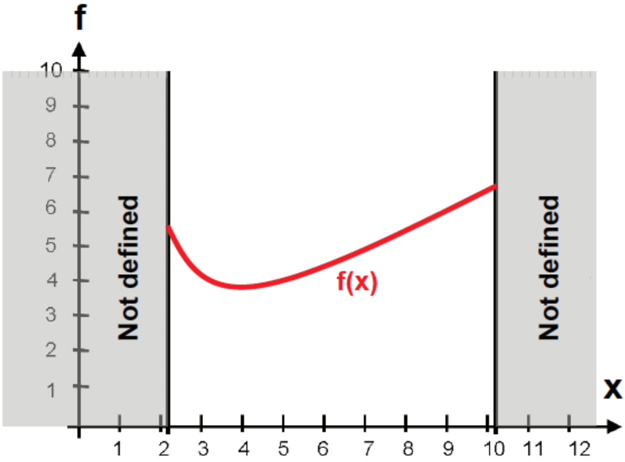
Common mistakes, i.e. false properties

- If f is non-convex, then f is concave   counter-ex: $f(x) = \sin(x)$
- If f is non-concave, then f is convex   counter-ex: $f(x) = \sin(x)$
- A function must be concave or convex   counter-ex: $f(x) = \sin(x)$
- A function cannot be concave and convex   counter-ex: $f(x) = x$

Ex 1: Convex functions not defined for all x

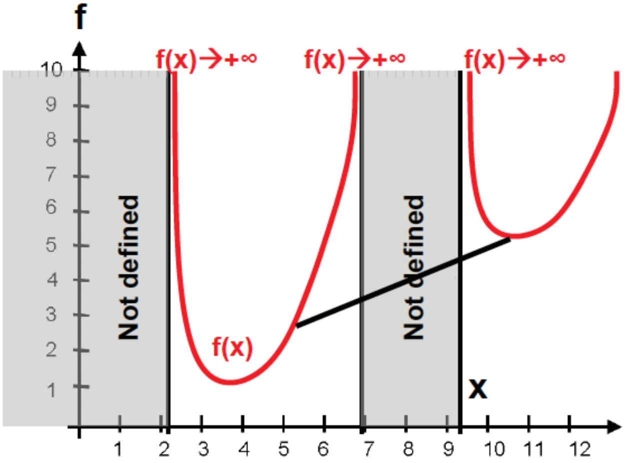


Ex 2: Convex functions not defined for all *x*

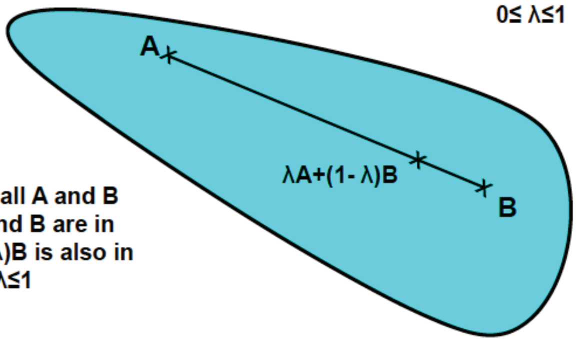


Ex 3: Domain is important

$f(x)$ is convex on $D_1 = \{x \in \mathbb{R} \mid 2.2 \leq x \leq 6.9\}$
$f(x)$ is convex on $D_2 = \{x \in \mathbb{R} \mid 7 \leq x \leq 9.2\}$
$f(x)$ is not convex on $D_3 = \{x \in \mathbb{R} \mid 2.2 \leq x \leq 9.2\}$



Convex Sets

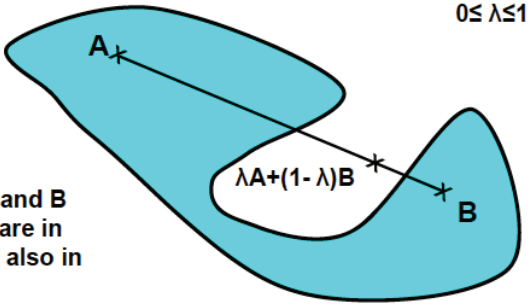


Definition:
Convex set: for all A and B in the set, if A and B are in the set, $\lambda A+(1-\lambda)B$ is also in this set, for $0\leq \lambda\leq 1$

This set is convex

Non-convex Sets

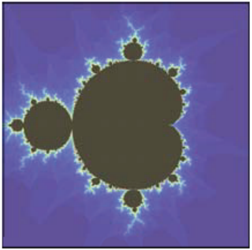
Definition:
Convex set: for all A and B in the set, if A and B are in the set, λA+(1- λ)B is also in this set, for 0≤ λ≤1



This set is not convex

Famous Convex Sets

Famous Non-convex Sets



Definitions of minimizers

Def'n (Global minimizer) : $x^* \in D$ is a global minimizer of f on D if

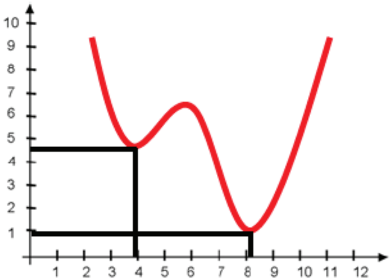
$$f(x^*) \leq f(x) \qquad \forall x \in D$$

in English: x^* minimizes f everywhere in D.

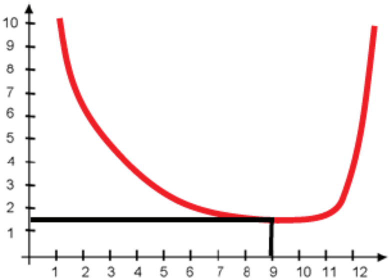
Def'n (Local minimizer) : $x^* \in D$ is a local minimizer of f on D if

$$\exists\, \epsilon > 0 \quad \text{s.t.} \quad f(x^*) \leq f(x) \qquad \forall x \in D \cap \{x \in \mathbb{R} \mid \|x - x^*\| < \epsilon\}$$

in English: x^* minimizes f locally in D.

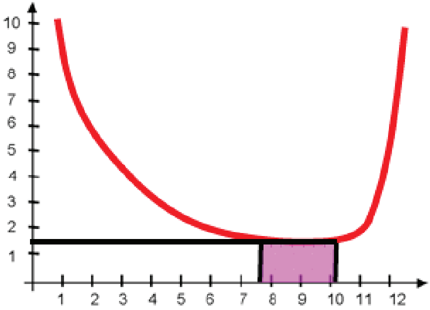


Local minimum might not be a global minimum

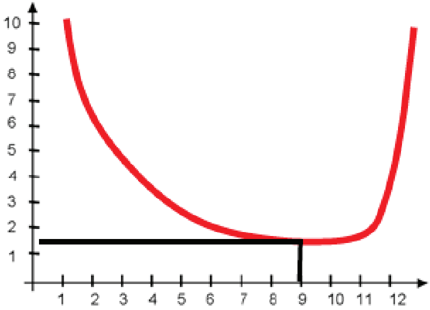


Local minimum is a global minimum

Uniqueness of minimizers



Minimum might not be unique



Local minimum is a global minimum

Q: Why should I care about convex functions, convex sets, and types of minimizers?

Convex Optimization Problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s. to} \quad & g(x) \leq 0 \end{aligned}$$

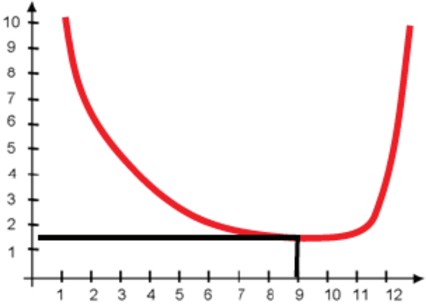
f is a convex function

$g(x) \leq 0$ encodes a convex set

Convex optimization problems are relatively "easy" to solve

- no analytical solution
- reliable and efficient algorithms
- many tricks for transforming problems into convex form
- surprisingly, many problems can be solved via convex optimization

Convex function



Local minimum is a global minimum

History of Convex Optimization

Theory: convex analysis: ca1900 - 1970

Algorithms:

- 1947: simplex algorithm for linear programming (Dantzig)
- 1960s: early interior-point methods (Fiacco & McCormick, Dikin,...)
- 1970s: ellipsoid method and other subgradient methods
- 1980s: polynomial-time interior-point methods for linear programming (Karmarkar 1984)
- late 1980s–now: polynomial-time interior-point methods for nonlinear convex optimization (Nesterov & Nemirovski 1994)

Applications:

- before 1990: mostly in operations research; few in engineering
- since 1990: many new applications in engineering (control, signal processing, communications, environmental analysis, climate analysis, structures, geoengineering, project management,...)

Additional Reading

Boyd & Vandenberghe, Chapters 2 and 3

Stephen Boyd Convex Optimization Lectures on YouTube Channel
`http://www.youtube.com/view_play_list?p=3940DD956CDF0622`
These have become extremely popular in recent years.

EE 227 - Convex Optimization with Prof. El Gaouhi