CE 191: Civil and Environmental Engineering Systems Analysis

LEC 12 : Barrier & Penalty Functions

Professor Scott Moura
Civil & Environmental Engineering
University of California, Berkeley

Fall 2014

Gradient Descent

- Does not explicitly account for constraints.
- Barrier and penalty functions approximate constraints by augmenting objective function $f(x)$

Consider constrained minimization problem

$$\begin{aligned} \min_{x} \quad & f(x), \\ \text{s. to} \quad & g(x) \leq 0 \end{aligned}$$

converted to

$$\min_{x} \quad f(x) + \phi(x; \varepsilon)$$

where $\phi(x; \varepsilon)$ captures the effect of constraints & is differentiable, thus enabling gradient descent

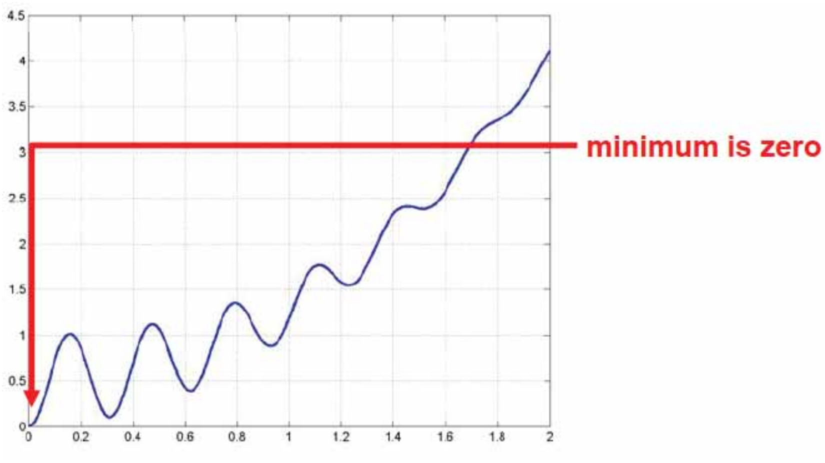
Two Methods for $\phi(x;\varepsilon)$: Barrier & Penalty Functions

- **Barrier Function:** Allow the objective function to increase towards infinity as x approaches the constraint boundary from inside the feasible set. In this case, the constraints are guaranteed to be satisfied, but it is impossible to obtain a boundary optimum.

- **Penalty Function:** Allow the objective function to increase towards infinity as x violates the constraints $g(x)$. In this case, the constraints can be violated, but it allows boundary optimum.

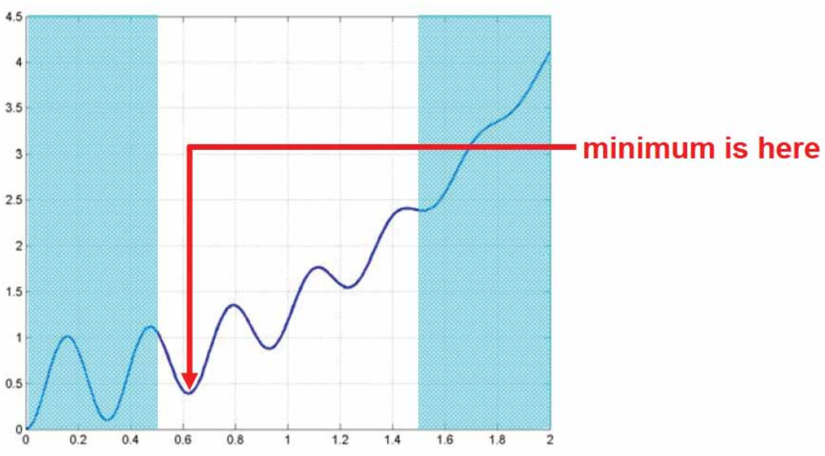
Constrained vs. Unconstrained Optimization

Example: find the optimum of the following function within the range $[0, +\infty)$



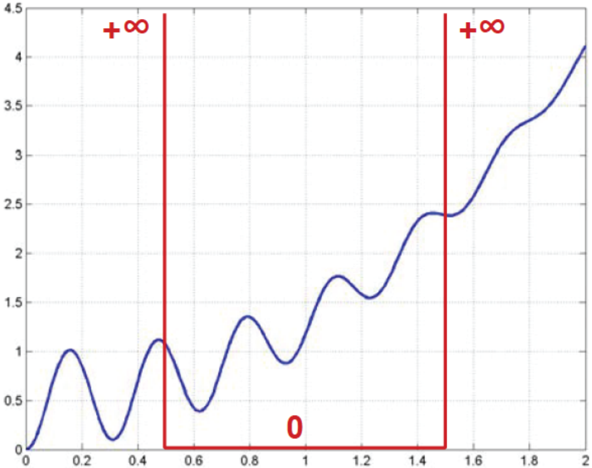
Constrained vs. Unconstrained Optimization

Example: find the optimum of the following function within the range $[0.5, 1.5]$



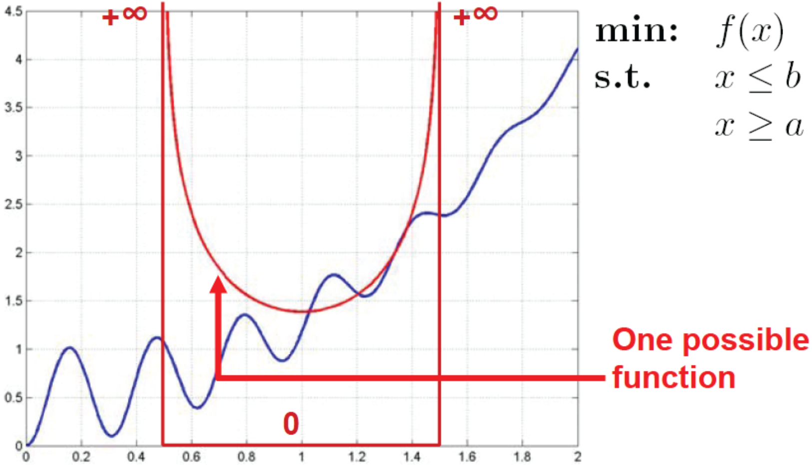
Main idea of barrier methods

Add a barrier function which is infinite outside of the constraint domain, i.e. $[a, b]$



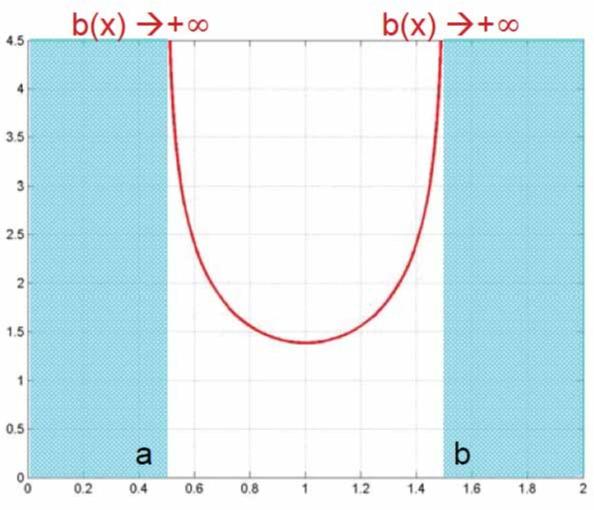
$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \leq b \\ & x \geq a \end{aligned}$$

In practice, such continuous and smooth functions do not exist, so they have to be approximated



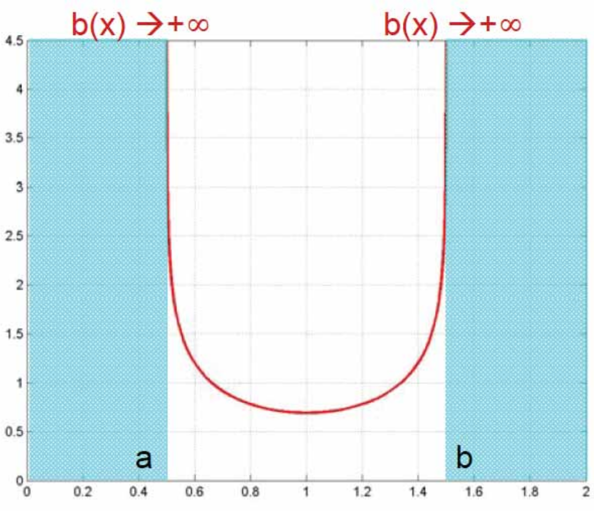
Logarithmic Barrier Function

$$b(x) = -\varepsilon \log((x-a)(b-x)), \qquad \varepsilon = 1$$



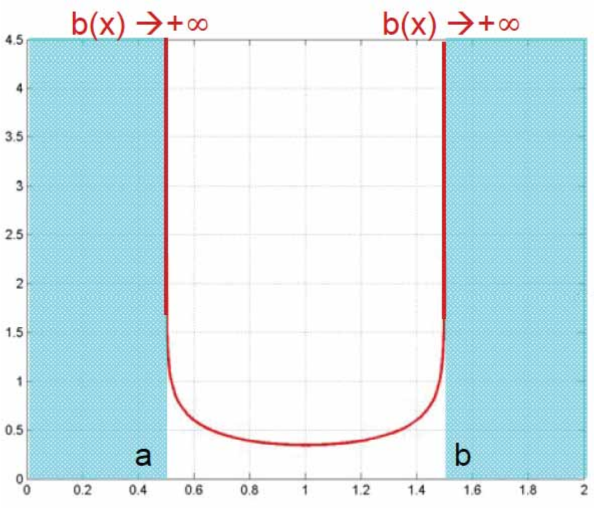
Logarithmic Barrier Function

$$b(x) = -\varepsilon \log((x-a)(b-x)), \qquad \varepsilon = 1/2$$



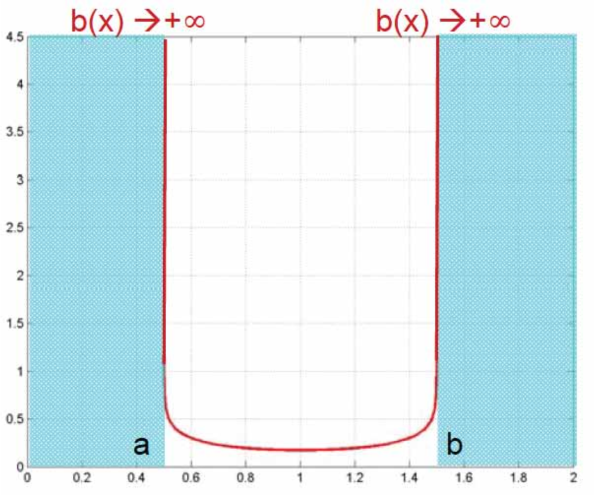
Logarithmic Barrier Function

$$b(x) = -\varepsilon \log((x-a)(b-x)), \qquad \varepsilon = 1/4$$



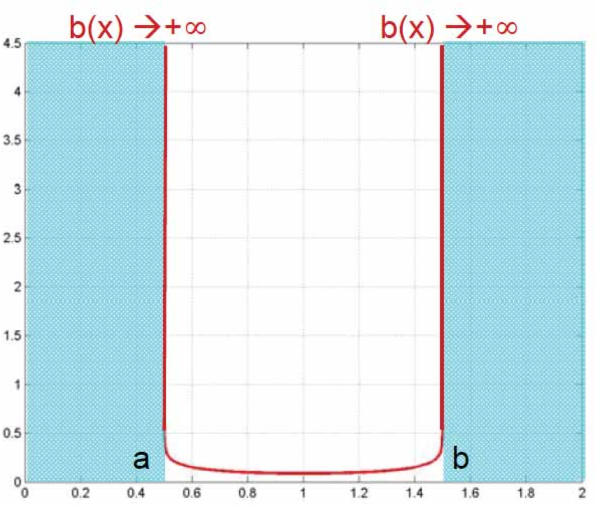
Logarithmic Barrier Function

$$b(x) = -\varepsilon \log((x-a)(b-x)), \qquad \varepsilon = 1/8$$



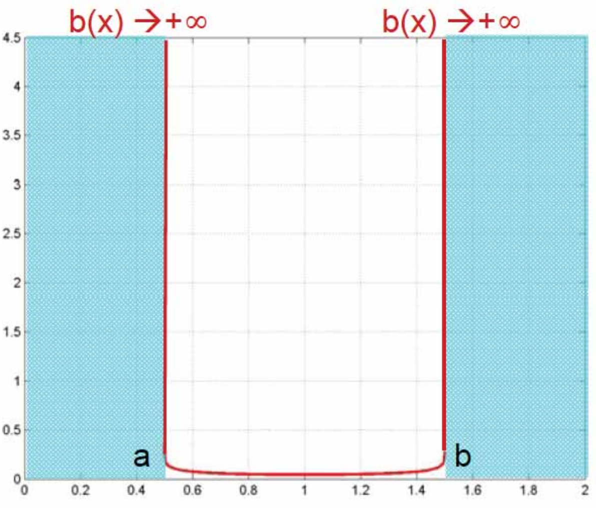
Logarithmic Barrier Function

$$b(x) = -\varepsilon \log((x-a)(b-x)), \qquad \varepsilon = 1/16$$



Logarithmic Barrier Function

$$b(x) = -\varepsilon \log((x-a)(b-x)), \qquad \varepsilon = 1/32$$



Utilization of Barrier Function

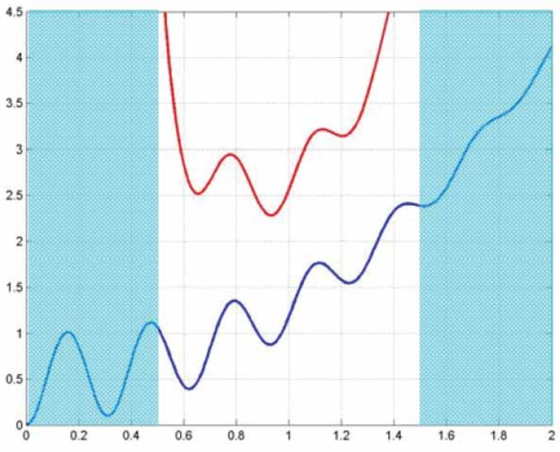
Add the barrier function $b(x)$ to the objective function $f(x)$

1. inside the constraint set, barrier ≈ 0
2. outside the constraint set, barrier is infinite

If the barrier is almost zero inside the constraint set, the minimum of the function and the augmented function are almost the same.

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1$

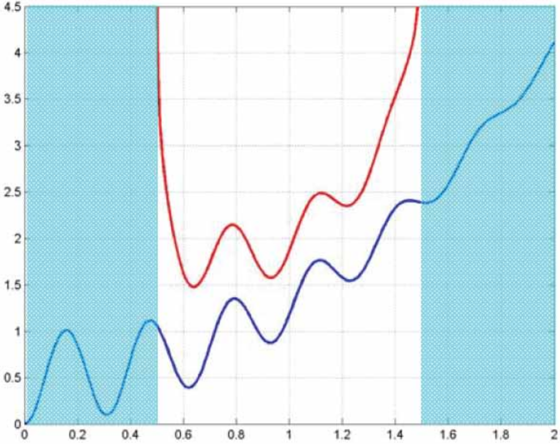


$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \in [a, b] \end{aligned}$$

$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \leq b \\ & x \geq a \end{aligned}$$

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/2$

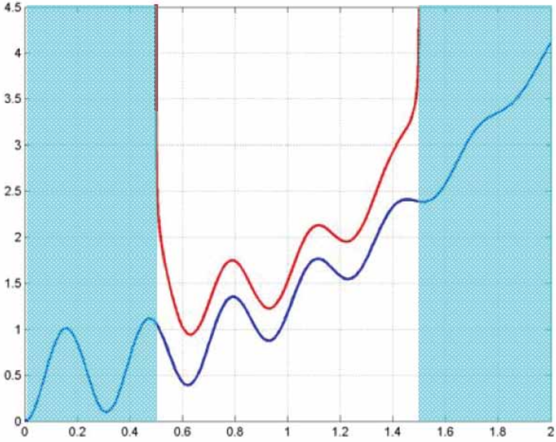


$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \in [a, b] \end{aligned}$$

$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \le b \\ & x \ge a \end{aligned}$$

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/4$

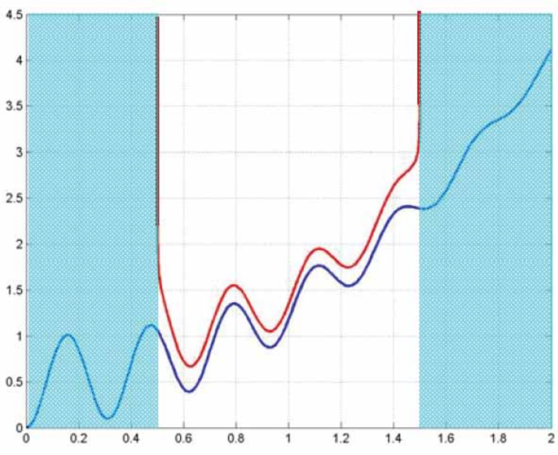


$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \in [a, b] \end{aligned}$$

$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \leq b \\ & x \geq a \end{aligned}$$

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/8$

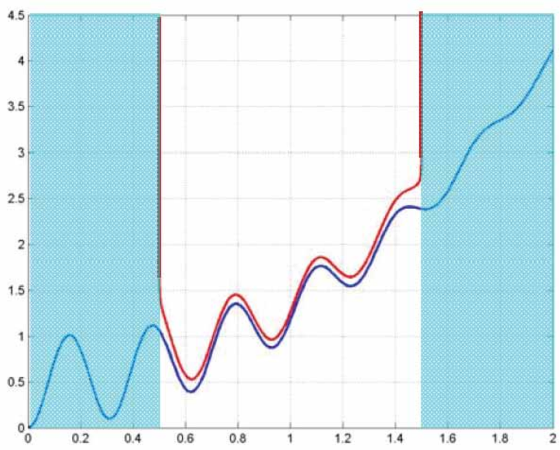


$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \in [a, b] \end{aligned}$$

$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \leq b \\ & x \geq a \end{aligned}$$

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/16$

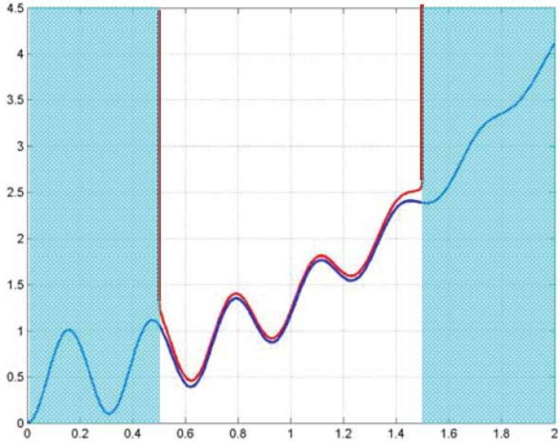


$$\begin{array}{ll} \textbf{min:} & f(x) \\ \textbf{s.t.} & x \in [a, b] \end{array}$$

$$\begin{array}{ll} \textbf{min:} & f(x) \\ \textbf{s.t.} & x \leq b \\ & x \geq a \end{array}$$

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/32$



$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \in [a, b] \end{aligned}$$

$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \leq b \\ & x \geq a \end{aligned}$$

Illustration of the convergence of the log barrier

Make a guess inside the constraint set.

Start with epsilon not too small.

repeat

- minimize the augmented function (using e.g. gradient descent)
- use the result as the guess for the next step
- decrease the log barrier

Until barrier is almost zero inside the constraint set

One can prove that the result of this method converges to a minimum of the original problem

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1$

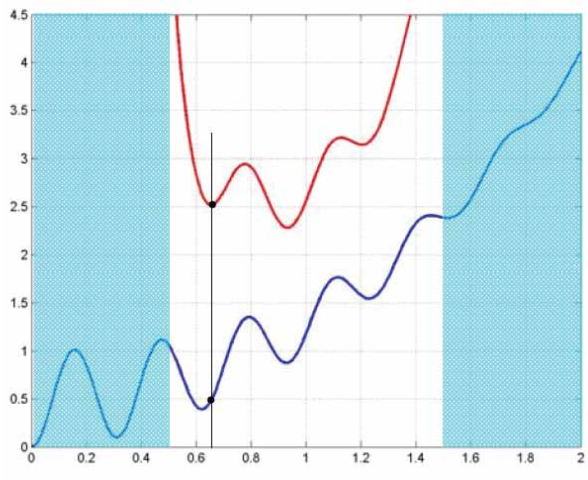
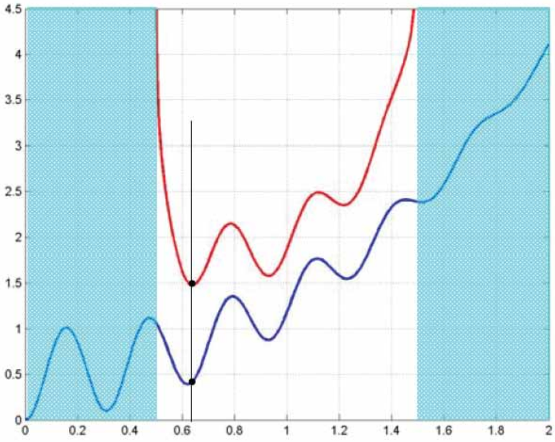


Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/2$

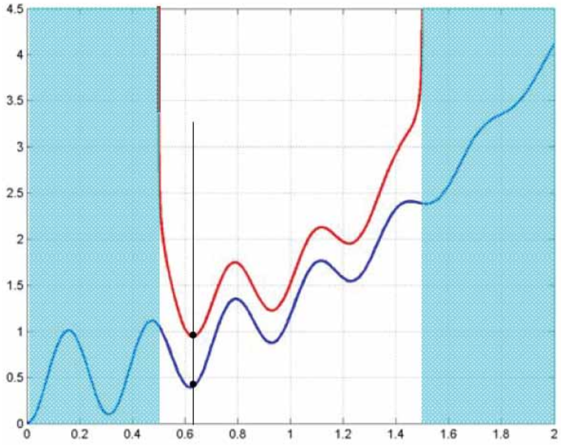


$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \in [a, b] \end{aligned}$$

$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \leq b \\ & x \geq a \end{aligned}$$

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/4$

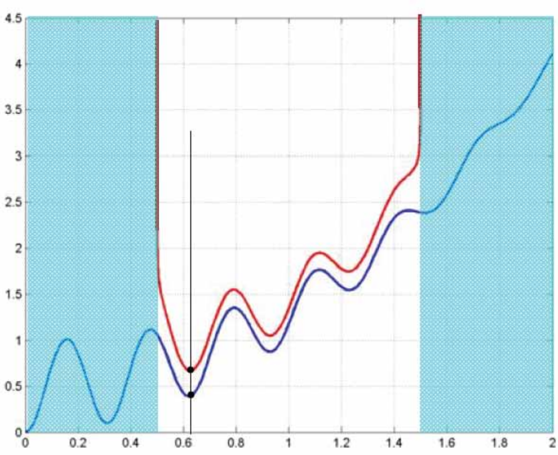


$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \in [a, b] \end{aligned}$$

$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \leq b \\ & x \geq a \end{aligned}$$

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/8$

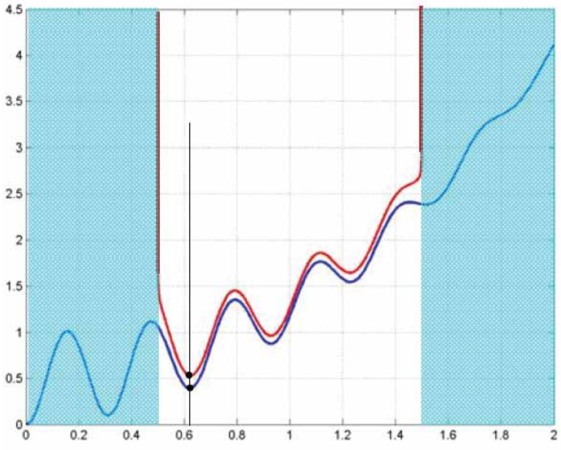


$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \in [a, b] \end{aligned}$$

$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \le b \\ & x \ge a \end{aligned}$$

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/16$

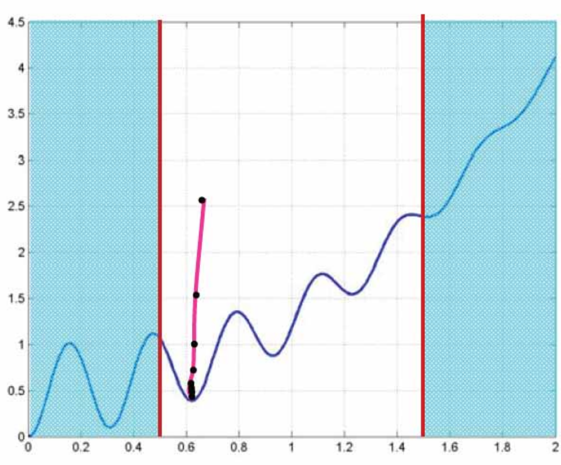


$$\begin{array}{ll} \textbf{min:} & f(x) \\ \textbf{s.t.} & x \in [a, b] \end{array}$$

$$\begin{array}{ll} \textbf{min:} & f(x) \\ \textbf{s.t.} & x \leq b \\ & x \geq a \end{array}$$

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/32$



$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \in [a, b] \end{aligned}$$

$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \leq b \\ & x \geq a \end{aligned}$$

Formal description of the algorithm

Start with epsilon not too small

repeat: solve

$$\begin{aligned} \min_x \quad & f(x) - \varepsilon b(x) \\ \text{s. to} \quad & \text{no constraints} \end{aligned}$$

use the result as the guess for the next step

decrease the log barrier $\varepsilon = \varepsilon = 2$, or similar

Until barrier is almost zero inside the constraint set

Generalization to multiple dimensions

Transformation of a constrained problem into an unconstrained problem

$$\min_{x} \quad f(x),$$
$$\text{s. to} \quad g(x) \leq 0$$

Introduce log barrier function

$$b(x) = \log(-g(x)) \tag{1}$$

Problem to solve becomes (in the limit ε goes to zero):

$$\min_{x} \quad f(x) - \varepsilon b(x)$$
$$\text{s. to} \quad \text{no constraints}$$

Penalty Function Method

Transformation of a constrained problem into an unconstrained problem

$$\begin{aligned} \min_{x} \quad & f(x), \\ \text{s. to} \quad & g(x) \leq 0 \end{aligned}$$

Introduce quadratic penalty function

$$\phi(x;\varepsilon) = \begin{cases} 0 & \text{if } g(x) \leq 0 \\ \frac{1}{2\varepsilon}(x - g(x))^2 & \text{otherwise} \end{cases} \tag{2}$$

Problem to solve becomes (in the limit ε goes to zero):

$$\begin{aligned} \min_{x} \quad & f(x) + \phi(x;\varepsilon) \\ \text{s. to} \quad & \text{no constraints} \end{aligned}$$

Additional Reading