

CE 191: Civil and Environmental Engineering Systems Analysis

LEC 14 : Introduction to Dynamic Programming

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Motivating Example: Traveling Salesman

What is the shortest path to loop through N cities?



[<http://www.informatik.uni-leipzig.de/meiler>]

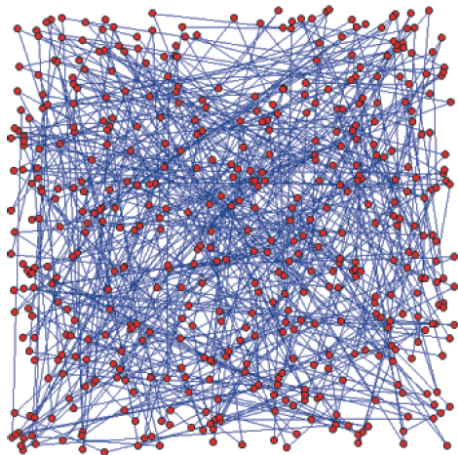


[<http://www.superbasescientific.com/>]

Traveling Salesmen

What is the shortest path to loop through N cities?

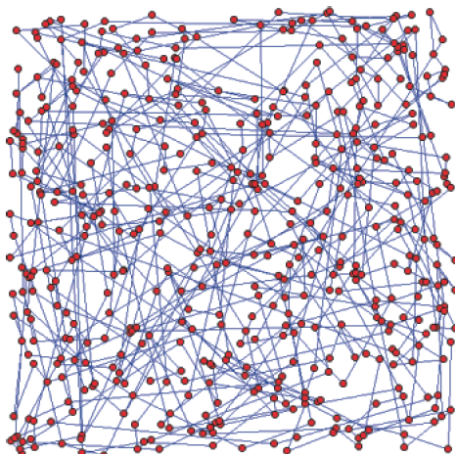
500 cities, **random** solution



Traveling Salesmen

What is the shortest path to loop through N cities?

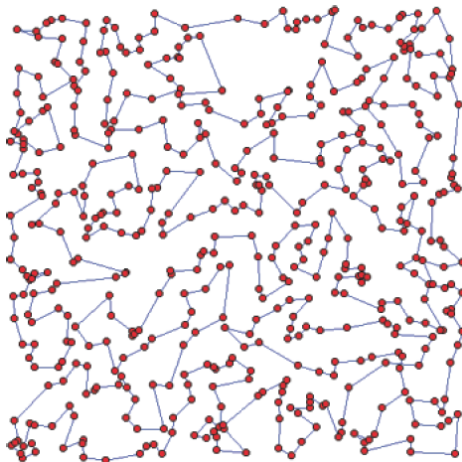
500 cities, a **better** solution



Traveling Salesmen

What is the shortest path to loop through N cities?

500 cities, **best** solution

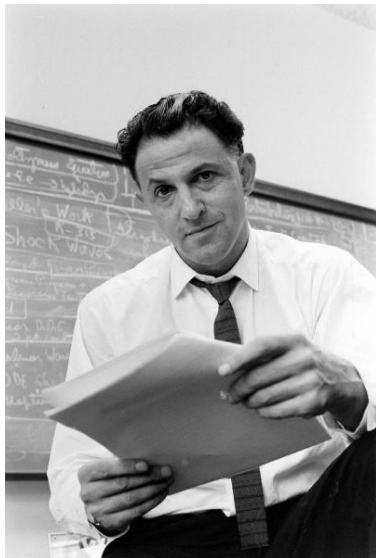


When to use DP?

- When decisions are made in stages
- Sometimes, decisions cannot be made in isolation.
- One needs to balance immediate cost with future costs.
- Curse of dimensionality, i.e. NP-hard problems

Applications

- Maps.
- Robot navigation.
- Urban traffic planning.
- Network routing protocols.
- Optimal trace routing in PCBs.
- HR scheduling and project management.
- Routing of telecommunications messages.
- Hybrid electric vehicle energy management.
- Optimal truck routing through given traffic congestion pattern.



Richard Bellman, Ph.D. | 1920 - 1984
University of Southern California
RAND Corporation

Coining “Dynamic Programming”

“I spent the Fall quarter (of 1950) at RAND. My first task was to find a name for multistage decision processes ... The 1950s were not good years for mathematical research. We had a very interesting gentleman in Washington named Wilson. He was Secretary of Defense, and he actually had a pathological fear and hatred of the word research. I’m not using the term lightly; I’m using it precisely. His face would suffuse, he would turn red, and he would get violent if people used the term research in his presence. You can imagine how he felt, then, about the term mathematical. The RAND Corporation was employed by the Air Force, and the Air Force had Wilson as its boss, essentially. Hence, I felt I had to do something to shield Wilson and the Air Force from the fact that I was really doing mathematics inside the RAND Corporation. What title, what name, could I choose? In the first place I was interested in planning, in decision making, in thinking. But planning, is not a good word for various reasons. I decided therefore to use the word “programming”. I wanted to get across the idea that this was dynamic, this was multistage, this was time-varying... Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities.”

Eye of the Hurricane: An Autobiography (1984)

Discrete-time system

$$x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \dots, N-1$$

k : discrete time index

x_k : state - summarizes current configuration of system at time k

u_k : control - decision applied at time k

N : time horizon - number of times control is applied

Additive Cost

$$J = \sum_{k=0}^{N-1} g_k(x_k, u_k) + g_N(x_N)$$

g_k : instantaneous cost - instantaneous cost incurred at time k

g_N : final cost - incurred at time N

Example 1: Inventory Control

Order items to meet demand, while minimizing costs

x_k stock available at the beginning of period k

u_k stock order and delivered immediately at the beginning of period k

d_k demand during the k^{th} period (assume deterministic)

Stock evolves according to

$$x_{k+1} = x_k + u_k - d_k$$

where negative stock corresponds to backlogged demand.

Three types of cost:

(a) $r(x_k)$ is penalty for positive stock (holding costs) or negative stock (shortage cost)

(b) The purchasing cost $c_k u_k$, where c_k is the cost per unit order at time k .

(c) Terminal cost $R(x_N)$ for excess stock or unfulfilled orders at time N .

Total cost:

$$J = \sum_{k=0}^{N-1} [r(x_k) + c_k u_k] + R(x_N)$$

Minimize cost by proper choice of $\{u_0, u_1, \dots, u_{N-1}\}$ subject to $u_k \geq 0$.

Principle of Optimality (in words)

Break the multistage decision problem into *subproblems*.

At time step k , assume you know optimal decisions for time steps $k + 1, \dots, N - 1$.

Compute best solution for current time step, and pair with future decisions.

Start from end. Work backwards recursively.

In the words of French researcher Kaufmann:
An optimal policy contains only optimal subpolicies.

Principle of Optimality (in math)

Define $V_k(x_k)$ as the optimal “cost-to-go” from time step k to the end of the time horizon N , given the current state is x_k .

Then the principle of optimality can be written in recursive form as:

$$V_k(x_k) = \min_{u_k} \{g_k(x_k, u_k) + V_{k+1}(x_{k+1})\}$$

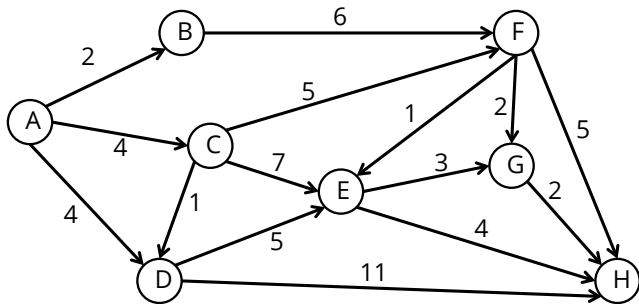
with the boundary condition

$$V_N(x_N) = g_N(x_N)$$

Admittedly awkward aspects:

- You solve the problem **backward!**
- You solve the problem **recursively!**

Example 2: Shortest Path Revisited



Let $V(i)$ be the shortest path from node i to node H.

Ex: $V(H) = 0$.

Let $c(i, j)$ denote the cost of traveling from node i to node j .

Ex: $c(C, E) = 7$.

$c(i, j) + V(j)$ is cost on path from node i to j , and then from j to H along shortest path.

Example 2: Shortest Path Revisited - Solution

Principle of optimality & boundary condition:

$$V(i) = \min_{j \in N_i^d} \{c(i, j) + V(j)\}$$

$$V(H) = 0$$

$$V(G) = c(G, H) + V(H) = 2 + 0 = 2$$

$$V(E) = \min \{c(E, G) + V(G), c(E, H) + V(H)\} = \min \{3 + 2, 4 + 0\} = 4$$

$$\begin{aligned} V(F) &= \min \{c(F, G) + V(G), c(F, H) + V(H), c(F, E) + V(E)\} \\ &= \min \{2 + 2, 5 + 0, 1 + 4\} = 4 \end{aligned}$$

$$V(D) = \min \{c(D, E) + V(E), c(D, H) + V(H)\} = \min \{5 + 4, 11 + 0\} = 9$$

$$\begin{aligned} V(C) &= \min \{c(C, F) + V(F), c(C, E) + V(E), c(C, D) + V(D)\} \\ &= \min \{5 + 4, 7 + 4, 1 + 9\} = 9 \end{aligned}$$

$$V(B) = c(B, F) + V(F) = 6 + 4 = 10$$

$$\begin{aligned} V(A) &= \min \{c(A, B) + V(B), c(A, C) + V(C), c(A, D) + V(D)\} \\ &= \min \{2 + 10, 4 + 9, 4 + 9\} = 12 \end{aligned}$$

Optimal Path: A \rightarrow B \rightarrow F \rightarrow G \rightarrow H

Revelle § 6.G, 13.A, 13.B

Denardo, Eric V. Dynamic programming: models and applications.
DoverPublications.com, 1982.

Bertsekas, Dimitri P. Dynamic programming and optimal control. Vol. 1. No. 2. Belmont: Athena Scientific, 1995.