CHAPTER 1: MODELING AND SYSTEMS ANALYSIS

1 Overview

The fundamental step in performing systems analysis and control design in energy systems is *mathematical modeling*. That is, we seek to write the ordinary differential equations (ODEs) that describe the physics of the particular energy system of interest. This process is highly non-trivial, and requires a careful combination of first principles (e.g. physics, chemistry, thermodynamics), experience, and creativity. Once these equations are properly derived, we then formulate the system dynamics into a so-called "state-space" form. The state-space form is the canonical template for analysis and control. State-space models can be divided into linear and nonlinear systems. We next focus on linear systems, and how they can be derived from nonlinear systems. The next and final fundamental concept is "stability". Stability, in rough terms, means the energy system does not "blow up" in some sense. In summary, this chapter is organized as follows:

1. Mathematical modeling of dynamic systems
2. State-space representations
3. Linear Systems
4. Stability

2 Mathematical Modeling of Dynamic Systems

Energy systems convert and store energy from a variety of physical domains, such as mechanical (e.g. flywheel), electrical (e.g. ultracapacitor), hydraulic (e.g. accumulator), chemical (e.g. gasoline), thermal (e.g. ice storage), (strong) nuclear (e.g. the energy binding a Uranium nucleus), economic (e.g. bank account) and more. As such, engineers and scientists require a common framework for describing and analyzing energy systems. This common framework is mathematics, and we refer to our description of the dynamic energy systems as a *mathematical model*. One must understand that a mathematical model is, at best, a surrogate for the physical system, whose precision is subject to the assumptions and requirements made by the energy systems engineer. To quote eminent statistician Dr. George E. P. Box (1919 – 2013): *"Essentially, all models are wrong, but some are useful."*

2.1 First Principles Modeling vs. Data-Driven Modeling

The process of translating an unstructured technical (or non-technical) energy system into a precise and clearly defined mathematical model is far from trivial. In many cases, a well-established set of models already exist. In other cases, a model must be generated for our purpose. Therefore, we desire a methodology for model generation.

Unfortunately, such a methodology cannot be "algorithmic" in the sense that it would offer a recipe that, when exactly followed, is guaranteed to produce the best possible model. The process of abstraction from a complex energy system to a mathematical model is simply not amenable to such a high degree of formalization. It is, in many instances, an "art form" that requires experience and intuition. Nevertheless, this section will provide some useful concepts for model generation.

The focus below will be on "control-oriented models," i.e., models which capture a system's main static and dynamic phenomena, without creating an excessive computational burden. The main reason for this requirement is that the models are assumed to be used repeatedly in numerical computations (for instance, when optimizing the system design or feedforward control signals) or in real-time loops (for instance, in feedback control systems). In simpler terms, we anticipate using these control-oriented models for planning and/or operation, where computational simplicity is desired.

A contrasting modeling paradigm is often pursued by scientists. That is, a scientist often studies the natural or man-made world to theorize mathematical relations that explain her observations. The more accurate and detailed the model, the better. In this course, we follow Albert Einstein's advice: *"Everything Should Be Made as Simple as Possible, But Not Simpler."* That is, the mathematical model should be sufficiently detailed to suit our specific energy management goals.

Energy system model synthesis is often based on physical first principles, for instance, the first and second laws of thermodynamics, the Lagrange equations in mechanics, Maxwell's equations in electromagnetism, or the Navier-Stokes law in fluid mechanics. Compared to data-driven methods (for instance, correlation methods), this approach has at least four major benefits:

- The models obtained are *explanatory*. In contrast to data-driven models, which seek to match input-output predictions to data, first-principle models explain the internal dynamics. This provides enhanced engineering insight into the energy system – a particularly nice property when high performance of safety is critical.
- The models obtained are, in principle, able to *extrapolate* the system behavior. That is, they can be useful beyond the operating conditions used in model validation.
- The models can be formulated even if the real system is not available (system still in planning phase or too dangerous / expensive to be used for experiments).
- Once such a mathematical model exists for a first system, the adaptation of that model to minor system modifications is relatively easy. Subsequent controller designs, which are based on the system model, can then be carried out (almost) automatically. This time-saving approach is critical for real-world application.

In today's data rich world, data-driven models (a.k.a. black-box, machine learning, or artificial intelligence models) have received renewed interest. Examples include artificial neural networks, deep learning networks, time series models, random forests, support vector machines, and more. Unlike first principles models, data-driven models make no attempt to model the internal features. Instead, they focus on matching the input-output behavior to observational data. Data-driven models have various advantages and disadvantages, including:

- Data-driven model synthesis generally does not require one to fundamentally understand the underlying process. This enables scalability and generalizability in a sense. However, you are exposed to errors or misuse.
- Data-driven models can predict processes that lack well-understood first principles. Examples include human behavior, such as advertisement click rates, building plug-load power consumption, mobility behaviors, and more. They can also predict behaviors that are difficult to model with first principles, such as wind power generation in the presence of wind gusts, solar power generation in the presence of clouds, etc.
- Data-driven models are trained and validated with data. As a result, there is no guarantee of accurate predictions outside the training data regime. Thus, *extrapolation* is a weakness.

Note, our modeling philosophy need not be binary, i.e. pick black or white modeling. This course promotes a "hybrid" modeling approach. Namely, utilize first principles when possible and combine with data-driven models when necessary. In any case, a skilled energy system engineer should have both techniques within their tool set.

2.2 Dynamical Systems Model

Figure 1 provides a block-diagram schematic of a generic dynamic system model that evolves in time t. The system, denoted by Σ, is characterized by a set of state variables $x(t)$. The state variables are influenced by the input variables $u(t)$ that represent the (controlled or uncontrolled) action of the system's environment on the system. The output variables $y(t)$ represent the observable or measurable aspects of the system's response.

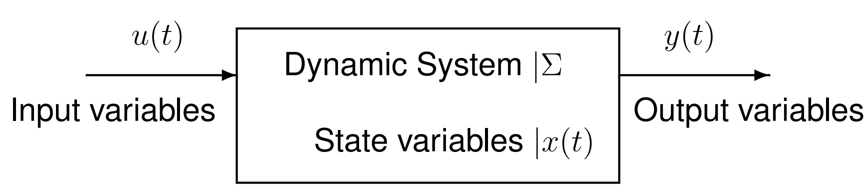

Figure 1: General dynamic system model.

In this chapter, we will refine the definition of this dynamical system model further. This definition, however, is sufficient to outline five potential applications of a mathematical model:

1. *Analysis.* Given a future trajectory of $u(t)$, $x(0)$ at the present, and the system model Σ, predict the future of $y(t)$. This use case is most commonly called "simulation" or "prediction". CH1 focuses on this topic.

2. *State Estimation.* Given a system Σ with time histories $u(t)$ and $y(t)$, find x that is consistent with Σ, u, y. This is *the monitoring problem*. That is, you cannot measure every state, yet you wish to monitor every state. You wish to synthesize an algorithm that fuses the model and measurable states to produce the unmeasurable states. CH2 focuses on this topic.

3. *System Design or Planning.* Given $u(t)$ and some desired $y(t)$, find Σ such that $u(t)$ acting on Σ will produce $y(t)$. Most engineering disciplines deal with design synthesis. Traditionally, one might create various physical prototypes to synthesize a desired system. This can, however, be very time-consuming and expensive. In fact, generating physical prototypes for a national electricity infrastructure, for example, is simply not possible. A mathematical model helps automate this process by virtually synthesizing designs. CH3 focuses on this topic.

4. *Model Identification.* Given time histories $u(t)$ and $y(t)$, usually obtained from experimental data, determine a model Σ and its parameter values that are consistent with u and y. Clearly, a "good" model is one that is consistent with a variety of data sets u and y. This is often called "system identification" in the control literature or simply "modeling" in machine learning. CH4 focuses on this topic.

5. *Control Synthesis.* Given a system Σ with current state $x(0)$ and some desired $y(t)$, find $u(t)$ such that Σ will produce $y(t)$. This is, put simply, the energy management problem. The idea is to construct a series of decisions that produce a desirable flow of energy through the system. CH5 focuses on this topic.

Using the dynamic system model in Fig. 1, you now have a formal framework to understand and unify the entire course. The remainder of the course notes are simply details.

2.3 Stocks and Flows

When modeling any physical system, we consider two main classes of objects:

- "stocks," for instance, thermal energy, kinetic energy, or information;
- "flows," for instance, heat, mass, etc. transferring between the stocks.

The notion of a stock is fundamental in system modeling, and only systems including one or more stocks exhibit dynamic behavior. For each stock there is an associated "level" variable that is a function of the stock's content The term "state variable" is used for this quantity in the systems and control community. The flows are typically driven by differences in the stock levels. Several examples will be given below.

Armed with the stock concept, we present general guidelines to formulate a control-oriented model. This encompasses (at least) the following seven steps:

1. precisely define the modeling objective;
2. define the system boundaries (what are the inputs? what are the outputs?);
3. identify the relevant stocks (of mass, energy, information,...) and the corresponding level (i.e. state) variables;
4. formulate the differential equations (conservation laws) for all relevant stocks

$$\frac{d}{dt}(\text{stock level}) = \sum \text{inflows} - \sum \text{outflows}; \tag{1}$$

5. formulate the (sometimes nonlinear) algebraic relations that express the flows between the stocks as functions of the level variable;
6. train the unknown system parameters from experimental data; and
7. validate the model against experimental data that was not used for model identification.

Remark 2.1. *The most common mistake in modeling is skipping Step 1 above. Without a precise objective, one has no criteria in which to evaluate which modeling features are necessary and which modeling features are superfluous. TIP: Write down your objective. Make it precise.*

Remark 2.2. *The second most common mistake is poorly executing Step 2 above, in the following sense. Within the system boundaries, you seek to formulate mathematical equations that describe the system's evolution. Outside the system boundaries is the environment. You do not seek to formulate equations that describe the environment. Instead, you seek to understand the environment's impact on the system. TIP: Ask yourself this question: What is within my system boundaries? What do I consider the environment?*

Example 2.1 (Room Thermal Dynamics in Buildings)**.** *Suppose you are an HVAC engineer and you wish to consider our lecture room as a dynamic thermal system (see Fig. 2). Eventually, you wish to regulate temperature in the room within a comfortable range, while minimizing electrical costs, subject to time-varying changes in ambient temperature and internal heat generation (e.g. from human occupancy or electrical devices). Our immediate objective is to formulate a mathematical model for this purpose. Perform Steps 1-5 of the modeling formulation procedure. How* *would the model be expanded to consider multiple lecture halls? What role does a thermostatically controlled HVAC unit play?*

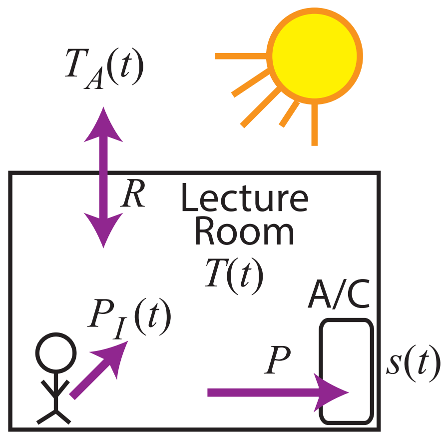

Figure 2: Schematic of Room Thermal Dynamics in Buildings.

Sample solution:

1. **Modeling Objective:** Our objective is to formulate a mathematical model that predicts how room temperature evolves, given interactions with the outside environment and an air conditioning (A/C) unit. This model will be used to design a controller that intelligently regulates room temperature to comfortable levels, while minimizing electricity costs.

2. **System Boundaries:** The system outputs include room temperature $T(t)$ and thermal power removed by the A/C unit $P(t)$. The system inputs include ambient environmental temperature $T_A(t)$, internal heat generation $P_I(t)$, and the A/C mode $s(t) \in \{0, 1\}$, corresponding to cooling or off.

3. **Stock(s):** The stock is thermal energy stored in the room. The level or state variable, indicating the amount of stored thermal energy, is $T(t)$.

4. **Conservation Law(s):** The room temperature $T(t)$ changes due to heat transfer with the outside environment and with the A/C unit. The First Law of Thermodynamics, in the form of Newton's Cooling Law, gives us:

$$C \frac{d}{dt} T(t) = \frac{1}{R} \left[T_A(t) - T(t) \right] - s(t) P + P_I(t) \tag{2}$$

   where parameter C is the room's thermal capacitance [kWh/°C] and R is the thermal resistance between the room and outside [°C/kW]. Consider the units. Note the left-hand side gives the time derivative of thermal energy inside the room – i.e. the rate at which the stock level changes. The right-hand side captures thermal power flowing in/out of the room – i.e. the flows in and out of the system.

5. **Relation between Flows and Level Variable:** These relations have already been expressed within the first and second terms on the right-hand side of (2). We can additionally compute the *electrical* power consumed by the A/C unit as:

$$P_{elec}(t) = s(t)P/\eta \tag{3}$$

where η represents the coefficient of performance (i.e. efficiency of electric-to-thermal power conversion).

- **Multiple Rooms:** If there are multiple rooms, then we would consider heat transfer between the rooms and between the rooms and environment.
- **Thermostatically Controlled HVAC:** A control law is an algorithm that actuates the HVAC unit through control signal $s(t)$, using measurements of room temperature $T(t)$. A typical thermostatic control law (assuming A/C only) is given mathematically by:

$$s(t) = \begin{cases} 0 & \text{if } T(t) \leq T_{sp} - \frac{\Delta}{2} \\ 1 & \text{if } T(t) \geq T_{sp} + \frac{\Delta}{2} \\ s(t-1) & \text{otherwise} \end{cases} \tag{4}$$

where T_{sp} is the set-point temperature and Δ is a deadband around the set-point.

Example 2.2 (Vehicle Dynamics)**.** *Suppose you are a mobility engineer, and you wish to analyze the energy consumption dynamics of a moving vehicle. Consider a vehicle (Fig. 3) of mass m, receiving traction force $F(t)$ from the engine, and a resistive force $k_0 + k_1 v^2(t)$ that includes rolling friction forces and viscous aerodynamic drag. Perform Steps 1-5 of the modeling formulation procedure.*

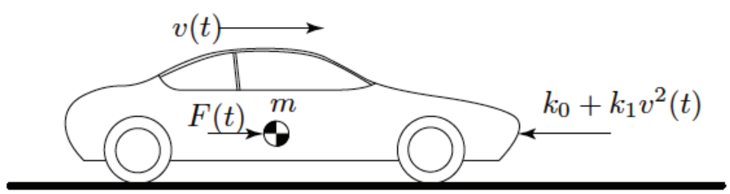

Figure 3: Free-body diagram of a moving vehicle (focused on longitudinal dynamics).

Sample solution:

1. **Modeling Objective:** Our objective is to predict energy consumption in a vehicle, given its velocity trajectory $v(t)$.

2. **System Boundaries:** The system outputs include traction force supplied by the engine $F(t)$, which we can map to fuel consumption given the appropriate relation. The system inputs include its velocity trajectory $v(t)$.

3. **Stock(s):** The stock is the kinetic energy of the vehicle. The level or state variable, indicating the amount of stored kinetic energy, is $v(t)$.

4. **Conservation Law(s):** The velocity $v(t)$ changes due to forces from the engine, rolling friction, and air drag. Newton's second law gives us:

$$m\frac{d}{dt}v(t) = F(t) - \left[k_0 + k_1 v^2(t)\right] \tag{5}$$

5. **Relation between Flows and Level Variable:** These relations have already been expressed on the right-hand side of (5).

Remark 2.3. *Students commonly make mistakes defining the system inputs and outputs. I encounter this difficulty when grading your midterms. The common mistake is that an input is a flow into the system from the environment, and an output is a flow from the system into the environment. That is wrong. The inputs & outputs refer to the flow of information (signal flow), not the flow of matter or energy. The system theoretic framework below provides clarification.*

2.4 System Theoretic Framework

After the ODEs have been written, usually based on first principles, we now think carefully about the unique roles of each variable or element, as detailed by the notation in Figure 4. To this end, we refine the dynamic system model framework defined in Fig. 1. All symbols in Table 4 may be scalar or interpreted as a column vector.

Figure 4: Mathematical Model Elements

Symbol	Description
x	State variable
u	Controllable inputs
w	Uncontrollable inputs
y	Measurable outputs
z	Performance outputs
θ	Parameters

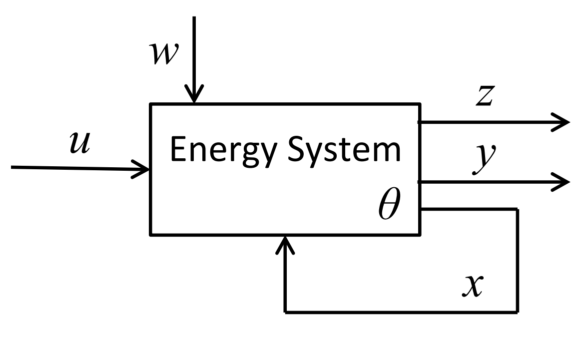

Figure 5: Block diagram of dynamic energy system.

The relationships between these variables are illustrated in Fig. 5. These six categories of variables serve distinct purposes in the dynamic energy system.

- The *state* x represents the dynamic condition of the energy system – in other words, the stock levels. Examples include battery state-of-charge, flywheel speed, power system frequency, room temperature, or financial account balance.
- The *controllable inputs* u represent a series of decisions we make to manage the energy system in a desirable way. These might include driver throttle position, thermostat set points, or wind turbine blade angles.
- The *uncontrollable inputs* w represent signals entering the energy system, which are not under our control. Examples include wind speed, solar irradiation, and traffic flow.
- The *measured outputs* y represent physical quantities that we measure with sensors. These might include voltage, vehicle speed, or classroom humidity.
- The *performance outputs* z represent quantities that we monitor, but may not measure directly with sensors. These might include fuel costs, time, or resources consumed.
- The *parameters* θ are a vector of scalar quantities which encapsulate physical properties of the system that, nominally, do not evolve with time. Examples include vehicle mass, thermal resistance of walls, or inductances of power transmission lines. These arise naturally from the physics or design of the system.

Example 2.3 (Flywheel Energy Storage)**.** Consider the flywheel shown in Fig. 6. The basic element of a flywheel is a rotating inertial mass connected to an electrical machine. Depending on which direction the machine applies torque, it converts energy between the electric and mechanical (kinetic energy) domains. Let us denote the flywheel inertia as I, the angular velocity as $\omega(t)$, a coefficient of friction as b, and the electric machine torque as $T(t)$. Then the equation of motion is governed by Euler's rotation equation:

$$I\dot{\omega}(t) = -b\omega(t) + T(t), \tag{6}$$

where $\dot{(\cdot)}$ denotes the derivative with respect to time. Note that $\omega(t)$ plays the role of state, $T(t)$ plays the role of controllable input, and the pair (I, b) play the role of parameters. If, in addition, we measure angular velocity then $\omega(t)$ is also a measurable output: $y(t) = \omega(t)$.

Example 2.4 (RLC Circuit)**.** We will now consider a simple series combination of three passive electrical elements: a resistor R, an inductor L, and a capacitor C, in series with a controllable voltage source V. Suppose we measure voltage across the resistor. This electrical system is known as an RLC circuit.

Since this circuit is a single loop, application of Kirchoff's current law (KCL) shows that the current is the same throughout the circuit at any given time, $i(t)$. Applying Kirchoff's voltage law

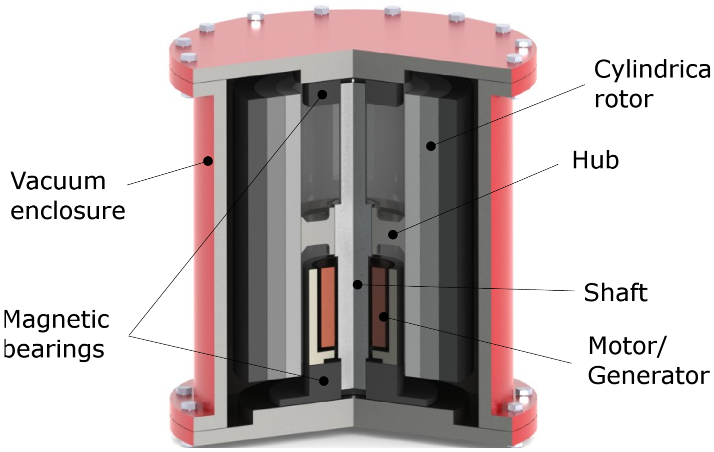

Figure 6: Main components of a typical flywheel.

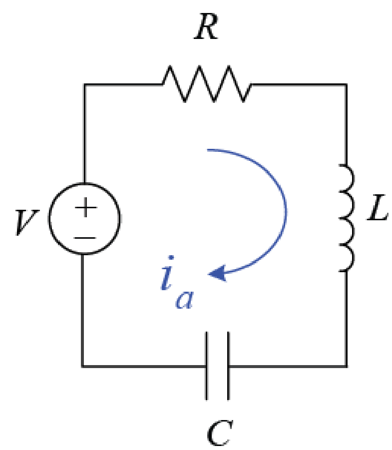

Figure 7: RLC circuit

(KVL) around the loop and using the sign conventions indicated in the diagram, we arrive at the following governing equation.

$$V(t) - L\frac{di}{dt}(t) - Ri(t) - \frac{1}{C}\int_0^t i(\tau)d\tau = 0. \tag{7}$$

Suppose we define $q(t)$ to be the capacitor charge, such that $q(t) = \int_0^t i(\tau)d\tau$. Then (7) can be written as,

$$V(t) - L\ddot{q}(t) - R\dot{q}(t) - \frac{1}{C}q(t) = 0. \tag{8}$$

Note that (8) contains a double derivative in time. We call this a "second-order" equation. In contrast, (6) from the flywheel energy storage example is a "first-order" system. In the present example, the pair $(q, \dot{q})$ is the state, V plays the role of controllable input, and triple (R, L, C) play the role of parameters. The measurable output is voltage across the resistor, given mathematically as:

$$y(t) = Ri(t) = R\dot{q}(t). \tag{9}$$

Example 2.5 (Automated Vehicle Platoon)**.** Consider a platoon of two automated long-distance freight trucks (i.e. a "micro-platoon"), as shown in Fig. 8. To reduce air drag, and therefore fuel consumption, we seek to maintain a close distance between the two vehicles without risking collision. Assume you can measure the distance between trucks, and may control the propulsion of each truck. Following the same procedure as Example 2.2, we write the equations of motion using Newton's second law for each vehicle:

$$\dot{x}_1(t) = v_1(t), \qquad \dot{x}_2(t) = v_2(t), \tag{10}$$

$$m_1\dot{v}_1(t) = F_1(t) - k_0 - k_1v_1^2(t) - F_w(t), \qquad m_2\dot{v}_2(t) = F_2(t) - k_0 - k_1v_2^2(t) - F_w(t), \tag{11}$$

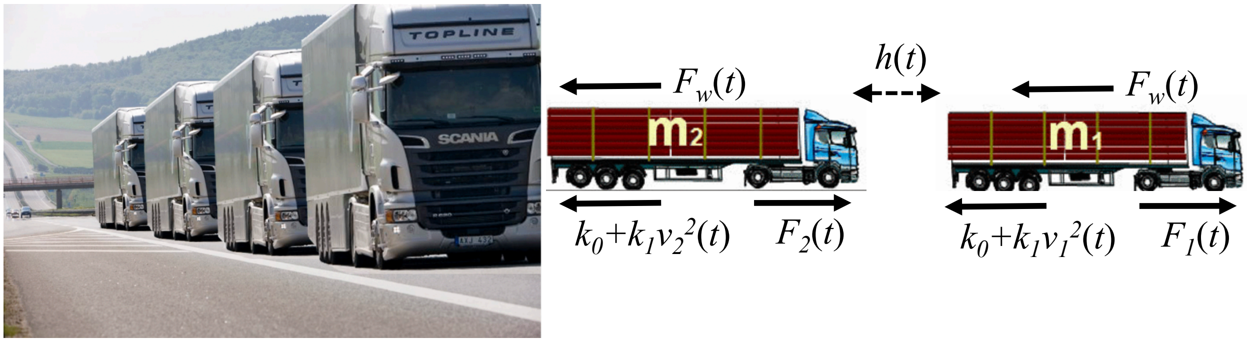

Figure 8: Micro-platoon of two long-distance freight trucks.

where x_i, v_i, represents the vehicle position and velocity respectively, and $i = 1, 2$ indexes the vehicles. Variables m_i, F_i are the vehicle mass and propulsion force, respectively. Coefficients k_0, k_1 are constants related to rolling friction and viscous aerodynamic drag. Finally, $F_w(t)$ is a random headwind force. Note that (10)-(11) contain four variables with time derivatives, x_1, x_2, v_1, v_2. This yields a fourth-order system.

Since we are specifically interested in the inter-vehicle distance, we define "headway" as $h(t) = x_1(t) - x_2(t)$. We can thus reformulate the model into a third-order system:

$$\dot{h}(t) = v_1(t) - v_2(t), \tag{12}$$

$$m_1 \dot{v}_1(t) = F_1(t) - k_0 - k_1 v_1^2(t) - F_w(t), \tag{13}$$

$$m_2 \dot{v}_2(t) = F_2(t) - k_0 - k_1 v_2^2(t) - F_w(t). \tag{14}$$

The system theoretic elements are:

Symbol	Description
$x(t) = [h(t), v_1(t), v_2(t)]^T$	State variables: headway, truck 1 velocity, truck 2 velocity
$u(t) = [F_1(t), F_2(t)]^T$	Controllable inputs: truck 1 propulsion, truck 2 propulsion
$w(t) = F_w(t)$	Uncontrollable inputs: Headwind force
$y(t) = [h(t), v_1(t), v_2(t)]^T$	Measurable outputs: headway & velocities
$z(t) = h(t)$	Performance outputs: headway - the variable we seek to regulate
$\theta = [m_1, m_2, k_0, k_1]^T$	Parameters: truck 1 mass, truck 2 mass, rolling friction & air drag coef.

Remark 2.4. *Clearly identifying the state variable is often difficult when you are first learning about dynamical systems modeling. Indeed, this concept is subtle. Here is a tip. Write down the differential equations first, before identifying the state variable. The variable which has a time derivative is very often the state variable.*

Next we discuss the canonical format for dynamical energy systems of any finite order, called the “state space representation”.

3 State-Space Representation

The state-space representation is a convenient and compact way to write the dynamics of energy systems. Put simply, it consists of a finite number of coupled first-order ordinary differential equations (ODEs)

$$\begin{aligned} \dot{x}_1(t) &= f_1\left(t, x_1(t), ..., x_n(t), u_1(t), ..., u_p(t)\right) \\ \dot{x}_2(t) &= f_2\left(t, x_1(t), ..., x_n(t), u_1(t), ..., u_p(t)\right) \\ \vdots \quad &\quad \vdots \\ \dot{x}_n(t) &= f_n\left(t, x_1(t), ..., x_n(t), u_1(t), ..., u_p(t)\right) \end{aligned} \tag{15}$$

where n represents the number of states, and p represents the number of controllable inputs. The uncontrollable input, w, can be added as well, but we suppress this for simplicity at this moment. We often use vector notation to write these equations in a compact form. Define

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_p(t) \end{bmatrix}, \quad f(t, x(t), u(t)) = \begin{bmatrix} f_1(t, x(t), u(t)) \\ f_2(t, x(t), u(t)) \\ \vdots \\ f_n(t, x(t), u(t)) \end{bmatrix} \tag{16}$$

Formally, we define vectors $x(t) : \mathbb{R}^+ \to \mathbb{R}^n$, $u(t) :\to \mathbb{R}^p$, $f(\cdot,\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$. Then we can re-write n first-order differential equations as a single $n-$dimensional first-order vector differential equation

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{17}$$

When q measurements are present, then we include another $q-$dimensional equation

$$y(t) = h(t, x(t), u(t)) \tag{18}$$

where $y(t)$ represents the q measured process outputs. Mathematically, $y(t) : \mathbb{R}^+ \to \mathbb{R}^q$. In summary, we refer to (17) as the state equation and (18) as the output equation. When the dynamical system does not have any control input, i.e. it runs *autonomously*, then we refer to it as an *autonomous system*. Mathematically, this is written,

$$\dot{x}(t) = f(t, x(t)) \tag{19}$$

An important fundamental concept in dynamical systems is *equilibria*. Roughly, the equilibrium of a dynamical system is the point in the state-space where the states remain constant, i.e. steady-state. It is defined mathematically as follows:

Definition 3.1. *(Equilibrium) Consider an autonomous dynamical system* $\dot{x}(t) = f(t, x(t))$*. The state* $x^{eq} \in \mathbb{R}^n$ *is called the equilibrium when it satisfies the following equation,*

$$0 = f(t, x^{eq}) \tag{20}$$

For non-autonomous systems, i.e. $\dot{x} = f(t, x, u)$*, consider a fixed input* $u^{eq} \in \mathbb{R}^p$*. The state* $x^{eq} \in \mathbb{R}^n$ *is called the equilibrium with respect to (w.r.t.) input* u^{eq} *when it satisfies the following equation,*

$$0 = f(t, x^{eq}, u^{eq}) \tag{21}$$

Remark 3.1. Energy systems do not always present themselves directly as a mathematical model in state space form, with their equilibria explicitly defined. More often, we arrive at a collection of ODEs and algebraic equations from first principles. However, we can usually carefully select the state variables to form a state-space representation and compute equilibria. This task arises in the analysis of many energy systems, including tokamak plasmas for thermonuclear fusion, multiphase fluid mechanics in oil & gas drillstrings, electrochemical battery models, electromagnetic models of AC induction motors in high-performance EVs, and more. We illustrate the computation of equilibria with the following example.

Example 3.1 (RLC Circuit Revisited)**.** Recall the RLC circuit studied in Example 2.4. The dynamics are described by the following second-order differential equation,

$$V(t) - L\ddot{q}(t) - R\dot{q}(t) - \frac{1}{C}q(t) = 0. \tag{22}$$

where we defined $\dot{q}(t) = i(t)$. Consequently, we can re-write these second-order dynamics in state-space form

$$\dot{q}(t) = i(t) \tag{23}$$

$$\dot{i}(t) = -\frac{1}{LC}q(t) - \frac{R}{L}i(t) + \frac{1}{L}V(t) \tag{24}$$

Note that the state vector is $x(t) = [q(t), i(t)]^T$ and the control input is $u(t) = V(t)$. Consequently, we can re-write the above equations in vector form

$$\frac{d}{dt}\begin{bmatrix} q(t) \\ i(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} q(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V(t) \tag{25}$$

or

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{26}$$

with matrices $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 1}$ appropriately defined. Now consider a fixed voltage input $V(t) = V^{eq}$. Then the equilibrium x^{eq} by definition must satisfy

$$0 = Ax^{eq} + BV^{eq} \tag{27}$$

Assuming A is invertible, $x^{eq} = -A^{-1}BV^{eq}$.

Remark 3.2. Equation (26) is an extremely important and special case of (17). It is called a "linear system." The significance of (26) is that, when energy systems can be described by this format, there exists a very large set of tools for system analysis and control design. When (26) cannot appropriately capture the energy system dynamics, then the set of available tools decreases dramatically. In the next section we formally introduce linear systems. Before discussing linear systems, let us explore another example of equilibria for dynamic systems.

Example 3.2 (Microplatoon)**.** *Recall the micro-platoon from Example 2.5. The dynamics are described by the following third-order system:*

$$\dot{h}(t) = v_1(t) - v_2(t), \tag{28}$$
$$m_1\dot{v}_1(t) = F_1(t) - k_0 - k_1 v_1^2(t) - F_w(t), \tag{29}$$
$$m_2\dot{v}_2(t) = F_2(t) - k_0 - k_1 v_2^2(t) - F_w(t). \tag{30}$$

Suppose we seek to find the steady-state forces F_1^{eq}, F_2^{eq} to achieve a steady-state headway of h^{eq} at a fixed speed of $v_1^{eq} = v_2^{eq}$. Assume a constant wind force of F_w^{eq}. The equilibrium $x^{eq} = [h^{eq}, v_1^{eq}, v_2^{eq}]^T$, by definition, must satisfy:

$$0 = v_1^{eq} - v_2^{eq}, \tag{31}$$
$$0 = F_1^{eq} - k_0 - k_1(v_1^{eq})^2 - F_w^{eq}, \tag{32}$$
$$0 = F_2^{eq} - k_0 - k_1(v_2^{eq})^2 - F_w^{eq}. \tag{33}$$

The steady-state input that achieves x^{eq} is $F_i^{eq} = k_0 + k_1(v_i^{eq})^2 + F_w^{eq}$, for $i = 1, 2$. Note the steady-state input is invariant with respect to the desired headway h^{eq}.

Example 3.3 (Pendulum, Section 1.2.1 of [1])**.** Consider the simple pendulum shown in Fig. 9, where L denotes the length of the rigid massless rod, m denotes the mass of the bob, and θ represents the rod's angle with the vertical axis. The pendulum is free to swing within a vertical two-dimensional plane. There are three forces acting on the bob. First, gravity acts in the downward vertical direction. Second, a tension force acts along the rod, toward the frictionless pivot. Third, a friction force resists the bob's motion through the medium, which we assume to be proportional to the bob's speed with coefficient of friction k. It is straight-forward to derive the equation of motion by (i) drawing a Cartesian coordinate system centered on the bob, (ii) rotating the axes to

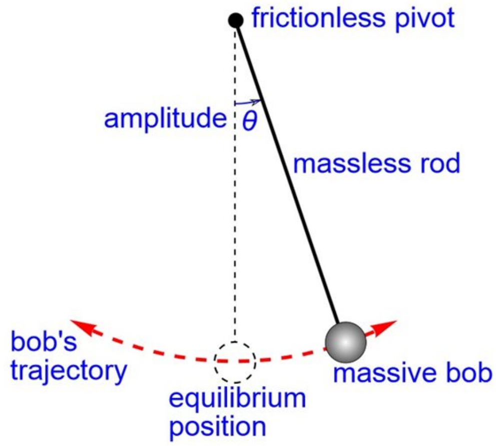

Figure 9: Pendulum.

be parallel and perpendicular to the rod, (iii) drawing a free-body diagram with forces aligned with the coordinate axis, (iv) applying Newton's second law:

$$mL\ddot{\theta}(t) = -mg\sin\theta(t) - kL\dot{\theta}(t) \tag{34}$$

This is a second-order system, similar to Example 3.1. Defining the state vector as $x(t) = [x_1(t), x_2(t)]^T = [\theta(t), \dot{\theta}(t)]^T$, we can re-write the second-order dynamics into a state-space from

$$\dot{x}_1(t) = x_2(t), \tag{35}$$

$$\dot{x}_2(t) = -\frac{g}{L}\sin x_1(t) - \frac{k}{m}x_2(t). \tag{36}$$

Unlike Example 3.1, this system is nonlinear due to the sinusoidal term in (36). To find the equilibrium points, we set $\dot{x}_1, \dot{x}_2 = 0$ and solve for x_1, x_2:

$$0 = x_2^{eq}, \tag{37}$$

$$0 = -\frac{g}{L}\sin x_1^{eq} - \frac{k}{m}x_2^{eq}. \tag{38}$$

The equilibrium points are located at

$$x_1^{eq} = n\pi, \quad \text{for } n = 0, \pm1, \pm2, \cdots, \qquad x_2^{eq} = 0. \tag{39}$$

From the physical description, the pendulum has only two unique equilibrium points $(0,0)$ and $(\pi, 0)$, which correspond to the downward and upward positions, respectively, with zero velocity. All other points are repetitions of these positions modulo an integer number of full swings from the reference position. Nevertheless there are, mathematically, an infinite number of equilibrium points. This example hints at the general fact that a nonlinear system may exhibit, one, multiple, or no equilibrium points.

Physical intuition suggests that equilibrium points $(0,0)$ and $(\pi,0)$ are quite distinct from each other. Namely, the pendulum can indeed rest at the $(0,0)$ equilibrium position. However, it can hardly remain at the $(\pi,0)$ position since any infinitesimally small disturbance will take the pendulum away from this position. The difference between these two equilibrium points is in their stability properties, a topic of enormous importance in practice and theory, which we study in Section 5.

4 Linear Systems

Linear systems are the foundation of systems and control engineering. In this section, we precisely define a linear system.

A *linear time-invariant (LTI)* system is described by the following state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{40}$$

$$y(t) = Cx(t) + Du(t) \tag{41}$$

where the state, control, and output have dimensions $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$, respectively. The matrices A, B, C, D are $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times p}$, $\mathbb{R}^{q\times n}$, and $\mathbb{R}^{q\times p}$, respectively. The system is said to be "time-invariant" when the matrices A, B, C, D are constant in time, e.g. Example 3.1

4.1 Linearization

No energy system is exactly linear and time-invariant. They are, in general, nonlinear and can be described by the nonlinear ODEs (17)-(18). Nonlinear system analysis is beyond the scope of this chapter (see [1]). However, linear approximations are often suitable for analysis and control design. Moreover, a wealth of tools exist for linear systems. Next, we describe how to approximate a nonlinear system by an LTI system. Recall Taylor's theorem:

Theorem 4.1. *(Taylor's Theorem) Let $k \geq 1$ be an integer. Let f be a nonlinear analytic function that maps scalars to scalars, i.e. $f : \mathbb{R} \to \mathbb{R}$, and is k times differentiable at the value $x = a$. Consider the power series expansion of $f(x)$ around $x = a$,*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_{k+1}(x) \tag{42}$$

where $R_{k+1}(x)$ is called the remainder term. Then infinite series converges to $f(x)$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{43}$$

if and only if $R_{k+1}(x) \rightarrow 0$ as $k \rightarrow \infty$. Note, this version of Taylor's theorem applies to functions of scalar reals, i.e. $f : \mathbb{R} \rightarrow \mathbb{R}$. Extensions exist for multivariable functions, i.e. $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

In our case, the utility of Taylor's theorem is a systematic method to approximate nonlinear systems by an LTI system. Consider the nonlinear ODE

$$\dot{x}(t) = f(x(t), u(t)), \tag{44}$$

and the equilibrium x^{eq} corresponding to u^{eq}. Then, we can define the *perturbation* from the equilibrium as $\tilde{x}(t) = x(t) - x^{eq}$ and $\tilde{u}(t) = u(t) - u^{eq}$. Graphically, we have simply translated the coordinate system so the origin is at (x^{eq}, u^{eq}). The ODE for $\tilde{x}(t)$ is then

$$\dot{\tilde{x}}(t) = \dot{x}(t) - \dot{x}^{eq} = f\left(\tilde{x}(t) + x^{eq}, \tilde{u}(t) + u^{eq}\right) - 0 \tag{45}$$

Now we apply Taylor series expansion, where the role of x in (42) is played by the pair $(\tilde{x}(t) + x^{eq}, \tilde{u}(t) + u^{eq})$ and the role of a in (42) is played by the pair (x^{eq}, u^{eq}),

$$\begin{aligned} \dot{\tilde{x}} &= f(x^{eq}, u^{eq}) + \frac{\partial f}{\partial x}(x^{eq}, u^{eq})\left(\tilde{x}(t) + x^{eq} - x^{eq}\right) + \frac{\partial f}{\partial u}(x^{eq}, u^{eq})\left(\tilde{u}(t) + u^{eq} - u^{eq}\right) && (46) \\ &\quad + R_2(\tilde{x}(t) + x^{eq}, \tilde{u}(t) + u^{eq}) && (47) \end{aligned}$$

Note that the first term satisfies the definition of an equilibrium and is therefore zero. Truncating the Taylor series expansion to remove the second order and higher remainder terms results in

$$\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(x^{eq}, u^{eq})\tilde{x}(t) + \frac{\partial f}{\partial u}(x^{eq}, u^{eq})\tilde{u}(t) \tag{48}$$

or

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t) \tag{49}$$

where $A = \frac{\partial f}{\partial x}(x^{eq}, u^{eq})$ and $B = \frac{\partial f}{\partial u}(x^{eq}, u^{eq})$. Matrices A and B are called the *Jacobians*.

To summarize, equation (49) represents the linearized dynamics of the nonlinear dynamic system (44), around equilibrium point (x^{eq}, u^{eq}). Very often, the linearized dynamics are sufficient to study energy systems, particularly around some desired operating point.

Example 4.1 (Magnetic Levitation, Example 4-11 of [2])**.** Figure 10 shows the diagram of a magnetic-ball suspension system. The objective of the system is to control the position of the steel ball by adjusting the current in the electromagnet through the input voltage $e(t)$. The differential equations for the system are given by

$$\begin{aligned} M\frac{d^2y(t)}{dt^2} &= Mg - \frac{i^2(t)}{y(t)}, && (50) \\ e(t) &= Ri(t) + L\frac{di(t)}{dt}. && (51) \end{aligned}$$

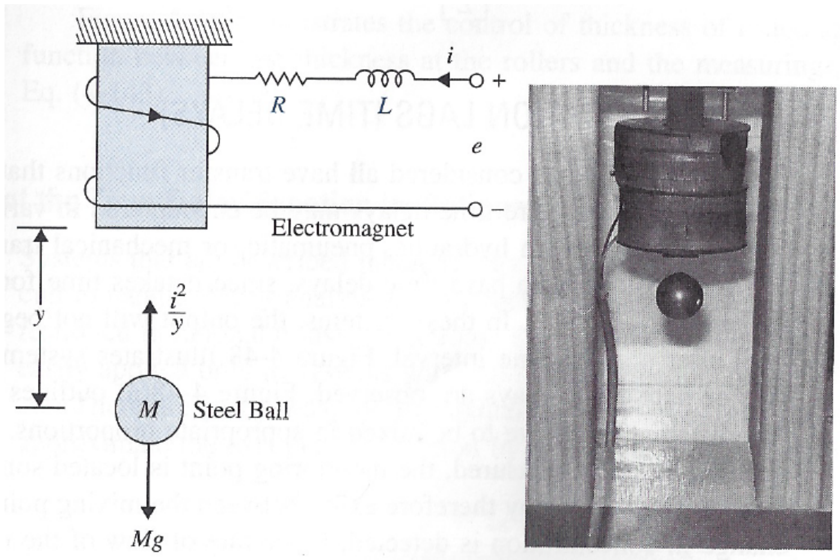

Figure 10: Magnetic ball suspension system.

where $e(t)$ is the input voltage, $y(t)$ is the ball position, $i(t)$ is the winding current, M is the ball's mass, g is gravitational acceleration, R is the winding resistance, and L is the winding inductance. Let us define the state variables as $x_1(t) = y(t)$, $x_2(t) = dy(t)/dt$, and $x_3(t) = i(t)$. The state equations of the system are

$$\frac{dx_1(t)}{dt} = x_2(t), \tag{52}$$

$$\frac{dx_2(t)}{dt} = g - \frac{1}{M}\frac{x_3^2(t)}{x_1(t)}, \tag{53}$$

$$\frac{dx_3(t)}{dt} = -\frac{R}{L}x_3(t) + \frac{1}{L}e(t). \tag{54}$$

Find the Equilibrium: Let us linearize the system around some desired equilibrium position $y^{des} = x_1^{eq} =$ constant. This could be $y^{des} = 10$cm, for example. The equilibrium values of the remaining states are,

$$x_2^{eq} = \frac{dx_1^{eq}}{dt} = 0, \qquad \frac{d^2 x_1^{eq}}{dt^2} = 0 \tag{55}$$

The equilibrium value of $x_3(t) = i(t)$ is determined by setting the LHS of (53) to zero, substituting $x_1^{eq} = y^{des}$, and solving for x_3^{eq}. We obtain

$$i^{eq} = x_3^{eq} = \sqrt{Mg \cdot y^{des}}. \tag{56}$$

Finally, the equilibrium value for the control input $e(t)$ can be found by setting the LHS of (54) to zero, substituting x_3^{eq} and solving for e^{eq}. This yields $e^{eq} = R\sqrt{Mg \cdot y^{des}}$. To summarize, the equilibrium is given by:

$$\begin{bmatrix} x_1^{eq} \\ x_2^{eq} \\ x_3^{eq} \end{bmatrix} = \begin{bmatrix} y^{des} \\ 0 \\ \sqrt{Mg \cdot y^{des}} \end{bmatrix}, \qquad e^{eq} = R\sqrt{Mg \cdot y^{des}}. \tag{57}$$

<u>Perturbation Variables:</u> Next we define perturbation variables:

$$\begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - x_1^{eq} \\ x_2(t) - x_2^{eq} \\ x_3(t) - x_3^{eq} \end{bmatrix}, \qquad \tilde{e}(t) = e(t) - e^{eq}. \tag{58}$$

This enables us to reformulate (52)-(54) as

$$\dot{\tilde{x}}_1(t) = f_1(\tilde{x}_1(t) + x_1^{eq}, \tilde{x}_2(t) + x_2^{eq}, \tilde{x}_3(t) + x_3^{eq}, \tilde{e}(t) + e^{eq}) \quad = \tilde{x}_2(t) + x_2^{eq}, \tag{59}$$

$$\dot{\tilde{x}}_2(t) = f_2(\tilde{x}_1(t) + x_1^{eq}, \tilde{x}_2(t) + x_2^{eq}, \tilde{x}_3(t) + x_3^{eq}, \tilde{e}(t) + e^{eq}) \quad = g - \frac{1}{M}\frac{(\tilde{x}_3(t) + x_3^{eq})^2}{\tilde{x}_1(t) + x_1^{eq}}, \tag{60}$$

$$\dot{\tilde{x}}_3(t) = f_3(\tilde{x}_1(t) + x_1^{eq}, \tilde{x}_2(t) + x_2^{eq}, \tilde{x}_3(t) + x_3^{eq}, \tilde{e}(t) + e^{eq}) \quad = -\frac{R}{L}\left[\tilde{x}_3(t) + x_3^{eq}\right] + \frac{1}{L}\left[\tilde{e}(t) + e^{eq}\right]. \tag{61}$$

Note that the equilibrium for the $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ system above is the origin $(0, 0, 0)$, by definition of the perturbation variables.

<u>Linearization:</u> Next we linearize the $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ system above around the origin. The resulting linearized system takes the standard form of (49), i.e. $\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t)$, with system matrices $A = \frac{\partial f}{\partial x}(x_1^{eq}, x_2^{eq}, x_3^{eq}, e^{eq})$ and $B = \frac{\partial f}{\partial u}(x_1^{eq}, x_2^{eq}, x_3^{eq}, e^{eq})$ defined as,

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}_{x_1 = x_1^{eq}, x_2 = x_2^{eq}, x_3 = x_3^{eq}, e = e^{eq}}, \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial e} \\ \frac{\partial f_2}{\partial e} \\ \frac{\partial f_3}{\partial e} \end{bmatrix}_{x_1 = x_1^{eq}, x_2 = x_2^{eq}, x_3 = x_3^{eq}, e = e^{eq}}. \tag{62}$$

After computing each partial derivative, plugging in the equilibrium expressions, and assembling the results into Jacobians A and B, we obtain:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(x_3^{eq})^2}{M(x_1^{eq})^2} & 0 & \frac{-2x_3^{eq}}{Mx_1^{eq}} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{g}{y^{des}} & 0 & -2\left(\frac{g}{My^{des}}\right)^{1/2} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix}. \tag{63}$$

Using mathematical model $\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t)$, with A, B defined above, we can study the magnetic ball dynamics around equilibrium position y^{des}. From here, we can use this LTI system description to, for example:

- Design control signal $u(t)$ to move the ball from one position to another;
- Estimate states x_1, x_2, x_3 from measurements of just x_1;
- Learn model parameters M, R, L.

Exercise 1. *Consider the pendulum dynamics from Example 3.3. Also consider a counter clockwise input torque $\tau(t)$ acting at the frictionless pivot. Also, assume a sensor measures the pendulum angle θ. Linearize the pendulum dynamics around the two equilibrium points $(0, 0)$, $(\pi, 0)$. What are the A, B, C, D matrices for each equilibrium?*

Exercise 2. *Consider the micro-platoon from Example 2.5, and state-space system (12)-(14). Assume a sensor measures the headway $h(t)$. Linearize the micro-platoon dynamics around the equilibrium point $x^{eq} = [h^{eq}, v_1^{eq}, v_2^{eq}]^T$. What is the corresponding equilibrium control input vector $y^{eq} = [F_1^{eq}, F_2^{eq}]^T$? What are the A, B, C, D matrices?*

5 Stability

Stability plays a central role in systems theory and engineering energy systems. In rough terms, stability means the energy system does not "blow up" in some sense. Think about thermal runaway in lithium-ion batteries, pressure buildup in steam generators, vibrations in wind turbines, or voltages in electrical distribution circuits. Clearly, guaranteeing stability is a desirable design criterion in the study of dynamic energy systems. The notion of stability goes beyond certifying against catastrophe. As shown in CH2 and CH3, it's also the critical concept for guaranteeing convergence of algorithms.

Within the dynamical systems literature, many different kinds of stability exist. In this chapter, we focus on the stability of equilibria in linear systems.

5.1 Definitions of Stability

Consider an autonomous LTI system, that is, a linear system with no control input,

$$\dot{x}(t) = Ax(t), \tag{64}$$

with an initial condition $x(0) = x_0$. We are now positioned to formally define stability.

Definition 5.1. *(Stability) The autonomous LTI system $\dot{x}(t) = Ax(t)$ is marginally stable if the solution, $x(t)$, to the ODE is bounded, i.e. $\max_t |x(t)| < \infty$ for all initial conditions x_0. The LTI system is asymptotically stable if the solution $x(t)$ converges asymptotically to zero, that is $\lim_{t\to\infty} x(t) = 0$ for all initial conditions x_0.*

Example 5.1. Consider the scalar autonomous LTI system and initial condition

$$\dot{x}(t) = 0, \qquad x_0 = 5 \tag{65}$$

The solution $x(t)$ is provided in Fig. 11. The solution remains bounded, indicating the system is *marginally stable*.

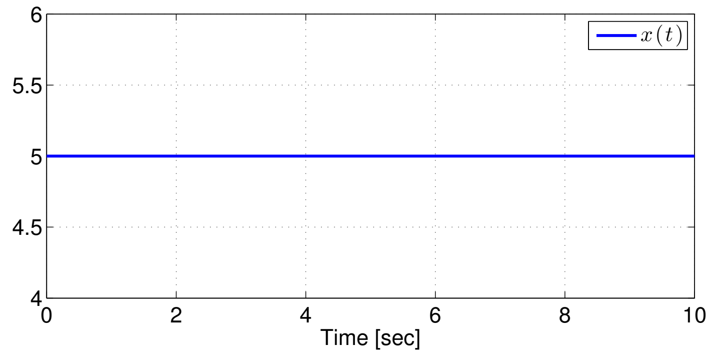

Figure 11: Solution to $\dot{x}(t) = 0, x_0 = 5$.

Example 5.2. Consider the second-order autonomous LTI system and initial condition

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \tag{66}$$

The solution $(x_1(t), x_2(t))$ is provided in Fig. 12. The solution oscillates, but remains bounded, indicating the system is *marginally stable*.

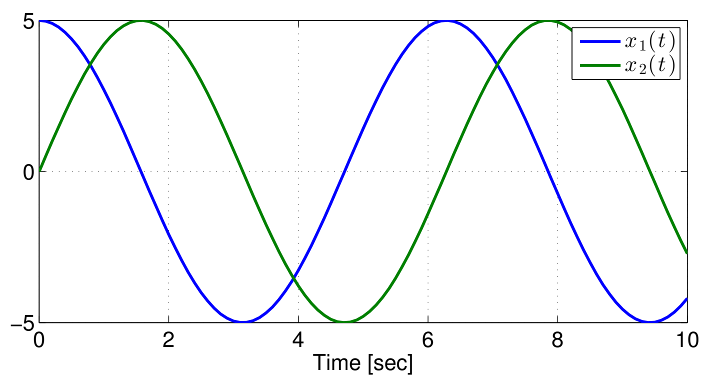

Figure 12: Solution to (66).

Example 5.3. Consider the slightly modified second-order autonomous LTI system and initial condition

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \tag{67}$$

The solution $(x_1(t), x_2(t))$ is provided in Fig. 13. In this case, the states clearly converge toward zero, indicating the system is *asymptotically stable*.

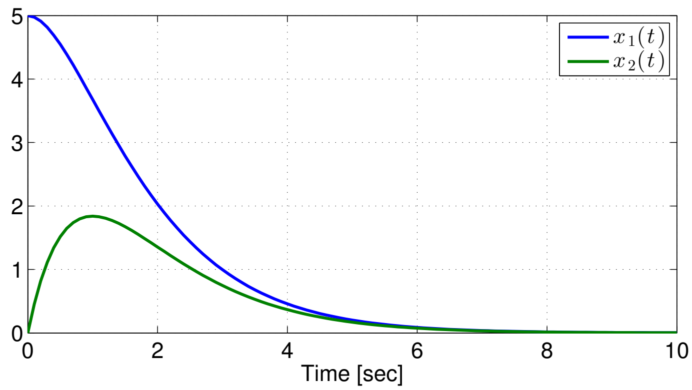

Figure 13: Solution to (67).

5.2 Tests for Stability

Although the previous definitions of stability are straightforward, it is undesirable to determine stability by solving the ODE directly. In practice, it would be more useful if a simple condition can be checked, without explicitly solving the ODE. For autonomous LTI systems, such a simple condition exists. It involves the eigenvalues of system matrix A. That is, denote λ_i, where $i = 1, ..., n$ the set of eigenvalues for A.

Theorem 5.1. *(LTI System Stability)*

1. *The LTI system $\dot{x}(t) = Ax$ is marginally stable if and only if all the eigenvalues of A contain zero or negative real parts, that is Re$[\lambda_i] \leq 0, \forall i$, and eigenvalues with zero real part are simple roots of the minimum polynomial of A*[1].

2. *The LTI system $\dot{x}(t) = Ax$ is asymptotically stable if and only if all the eigenvalues of A have strictly negative real parts, that is Re$[\lambda_i] < 0, \forall i$.*

Let us revisit the previous three examples, to demonstrate this condition.

Example 5.4. Consider the scalar autonomous LTI system and initial condition

$$\dot{x}(t) = 0, \qquad x_0 = 5 \tag{68}$$

[1] This last condition involving simple roots is technical. For interested readers, refer to Theorem 5.4 of [7]

The eigenvalues of the system matrix, which is trivially zero in this case, is zero. Consequently the system is marginally stable.

Example 5.5. Consider the second-order autonomous LTI system and initial condition

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \tag{69}$$

The eigenvalues of the system matrix are $\pm 1j$. The real parts are zero, indicating the system is marginally stable.

Example 5.6. Consider the slightly modified second-order autonomous LTI system and initial condition

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \tag{70}$$

The eigenvalues of the system matrix are $-1, -1$, implying the system is asymptotically stable.

Example 5.7. In this example, we illustrate a linear system with two eigenvalues at the origin that is, in fact, unstable. Consider the autonomous LTI system and initial condition

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{71}$$

From inspection, it is easy to verify the solution is given by: $x_1(t) = t; x_2(t) = 1$. Clearly, the system is unstable because $x_1(t) \to \infty$ as $t \to \infty$. The eigenvalues of the system matrix are $0, 0$, i.e. $\lambda(A) = 0$ with multiplicity of two. Consequently, Case 1 of Theorem 5.1 does NOT apply.

Exercise 3. *Consider the pendulum dynamics from Example 3.3 and the two equilibrium points $(0,0)$, $(\pi,0)$. Linearize the dynamics around each equilibrium point. Determine the system matrix A for each linearized system. Determine the stability properties. What are the eigenvalues?*

Exercise 4. *Consider the Room Thermal Dynamics Example 2.1. What are the equilibria when the HVAC unit is on $s = 1$ and off $s = 0$, for fixed ambient temperature T_A^{eq}? What are the eigenvalues in each case? Determine the stability properties.*

Exercise 5. *Consider the the Magnetically Levitated Ball from Example 4.1 with parameter values $M = 1$kg, $L = 100$mH, R = 10 mΩ. Determine a condition for the ball position x_1^{eq} in which the dynamics are asymptotically stable.*

Exercise 6. *Consider the RLC circuit from Example 2.4. Derive a condition on the parameter R, in terms of parameters L, C, that ensure asymptotic stability of the equilibrium corresponding to constant voltage input V^{eq}.*

5.3 Stability Analysis Examples for Nonlinear Systems

Example 5.8 (Thermal Runaway in Batteries)**.** In this example, we are going to demonstrate how stability analysis enables us to understand why lithium-ion batteries may experience thermal runaway (c.f. the Boeing 787 battery fire in 2013, the Samsung Galaxy Note 7 fires in 2016, etc.).

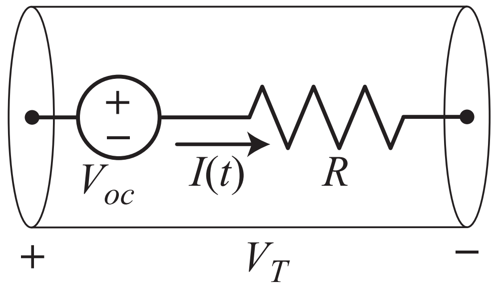

Figure 14: An equivalent circuit model for a battery cell, incorporating a constant voltage source in series with a temperature dependent resistor. The cell temperature $T(t)$ evolves due to heat transfer with the ambient T_∞ and ohmic heat generation from the resistor, as depicted by the dotted lines.

Consider an equivalent circuit model for the battery's electrical dynamics, as shown in Fig. 14. Kirchoff's voltage law yields

$$V_T = V_{oc} - R(T)I(t) \tag{72}$$

where V_T is the voltage at the battery's terminals, V_{oc} is the so-called "open-circuit" voltage, and $I(t)$ is the electric current withdrawn from the cell (assumed positive). Both V_T and V_{oc} are assumed constant. The term $R(T)I(t)$ represents the ohmic voltage drop due to internal resistance. The crucial modeling feature is that internal resistance $R(T)$ is a decreasing function of temperature T. That is, the ohmic losses become less as the battery temperature increases. Mathematically, $R'(T) < 0$.

A simple lumped thermal dynamics model is given by:

$$C_T \frac{d}{dt} T(t) = h\left[T_\infty - T(t)\right] + R(T)I^2(t) \tag{73}$$

The state is cell temperature $T(t)$. The parameters are as follows: C_T is the thermal heat capacity [J/K]; T_∞ is the ambient temperature - assumed fixed in this example; h is the heat transfer coefficient [W/K]; and the RI^2 term represents internal ohmic heat generation [W].

Using (72), we can find an explicit expression for electric current $I(t)$ in terms of temperature $T(t)$: $I(t) = \left[V_{oc} - V_T\right]/R(T(t))$. Substituting this expression into (73) yields

$$C_T \frac{d}{dt} T(t) = h\left[T_\infty - T(t)\right] + \frac{\left[V_{oc} - V_T\right]^2}{R(T(t))} \tag{74}$$

Note this ODE is nonlinear in state variable $T(t)$, due to the $1/R(T)$ term. Let us now provide intuition for the ODE above. The first term, which represents heat transfer with the environment, is stabilizing. Namely, it forces $T(t)$ toward T_∞, regardless of their relative values. The second term, which represents ohmic heat generation, is de-stabilizing. It is always positive. Worst yet, as $T(t)$ increases, then $R(T)$ decreases and the entire second term increases further. If this positive feedback loop overpowers the first term, then we experience thermal runaway.

Next we make this intuition mathematically precise with stability analysis. First, we find the equilibrium temperature T^{eq} using Definition 3.1:

$$0 = h\,[T_\infty - T^{eq}] + \frac{[V_{oc} - V_T]^2}{R(T^{eq})}. \tag{75}$$

This is a nonlinear equation in T^{eq} that can be solved numerically, given parameters values h, T_∞, V_{oc}, V_T and function $R(\cdot)$. Next, we linearize the dynamics (74) around equilibrium point $T = T^{eq}$. Define perturbation state variable: $\tilde{T}(t) = T(t) - T^{eq}$. Linearizing around this equilibrium, it is straight forward to show the resulting system takes the form of (49), i.e. $\dot{\tilde{T}}(t) = A\tilde{T}(t)$, with system matrix A defined as,

$$A = -\frac{h}{C_T} - \frac{[V_{oc} - V_T]^2}{C_T \cdot R(T^{eq})^2} \cdot R'(T^{eq}). \tag{76}$$

Theorem 5.1 tells how the eigenvalues of A relate to stability. Since A is scalar in this case, we need only worry about the sign of A. The first term of A is negative - creating a stabilizing effect. The second term of A is positive. Recall that $R'(T) < 0$ and $V_{oc} - V_T$ is positive. Consequently, we arrive at the following stability analysis conclusion:

- **Thermal Runaway:** The battery thermal dynamics (74) are unstable around equilibrium temperature T^{eq} defined by (75) if $-\frac{[V_{oc}-V_T]^2}{C_T \cdot R(T^{eq})^2} \cdot R'(T^{eq}) > \frac{h}{C_T}$.
- **Thermal Stability:** The battery thermal dynamics (74) are marginally stable or stable i.s.L. around equilibrium temperature T^{eq} defined by (75) if $-\frac{[V_{oc}-V_T]^2}{C_T \cdot R(T^{eq})^2} \cdot R'(T^{eq})$ in equal to or less than $\frac{h}{C_T}$, respectively.

Some simulations below illustrate...

Example 5.9 (Coupled Oscillators - A concept for understanding Power System Voltage Collapse)**.** In this example, we explore the essence of voltage collapse in power systems via coupled oscillators. Coupled oscillators, which are more fundamental and broad than power systems, are a simple yet rich model for conceptually understanding the dynamics of synchronous generators in power networks. This example is inspired by the work of Florian Döfler, John Simpson-Porco, and Francesco Bullo [3].

The power grid is composed of rotating masses (i.e. generators) distributed and electrically coupled across a network, as shown in the line diagram in Fig. 15(a). Circles indicate generators

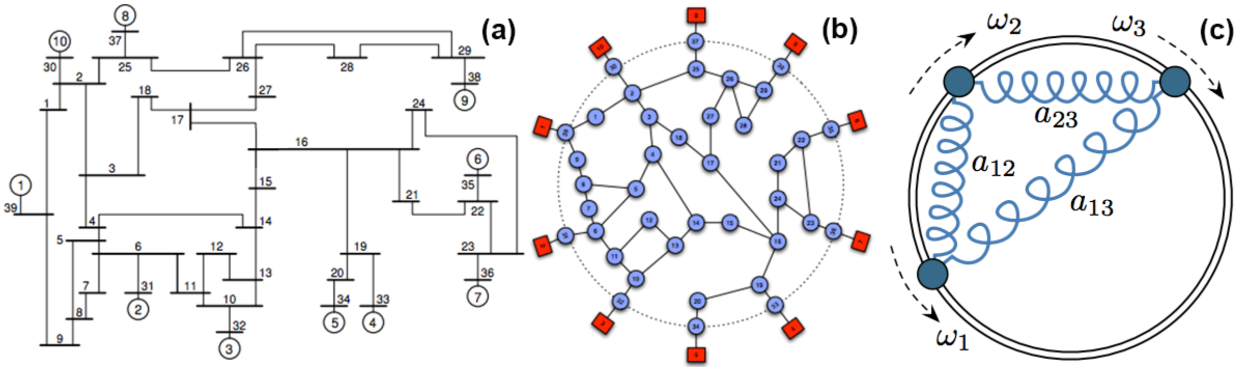

Figure 15: (a) Power system line diagram with synchronous generators (circles) and busses. (b) A mechanical analog would be oscillators interconnected by springs & dampers. The red squares are generators; the blue circles are busses. (c) A toy coupled oscillator with three elements.

and thick horizontal lines represent busses with loads. A mechanical analog for the transient dynamics of the interconnected synchronous generators and loads are a network of osccilators in Fig. 15(b). For simplicity of exposition, we consider three massless oscillators on a frictionless ring in Fig. 15(c). Each oscillator rotates around the ring with natural frequency ω_i. Each pair of oscillators is coupled by a spring, with spring constant a_{ij}. The coupled oscillator model, known as Kuramoto oscillator, is:

$$\dot{\theta}_i(t) = \omega_i - \sum_{j=1}^{N=3} a_{ij} \sin(\theta_i(t) - \theta_j(t)), \qquad i = 1, 2, 3 \tag{77}$$

We are interested in understanding the stability properties of these coupled oscillators. This analysis is the first step toward understanding voltage collapse in large-scale power systems and even microgrids.

First, we re-define the state vector as $\theta_{ij}(t) = \theta_i(t) - \theta_j(t)$. Note that the absolute angular displacement of the oscillators is not of interest. Instead, we are interested in understanding the relative angular displacement between two oscillators. Consequently, we derive the relative angular displacement dynamics $\dot{\theta}_{ij}(t) = \dot{\theta}_i(t) - \dot{\theta}_j(t)$, which yields:

$$\dot{\theta}_{ij}(t) = \omega_i - \sum_{k=1}^{N=3} a_{ik} \sin(\theta_i(t) - \theta_k(t)) - \omega_j + \sum_{k=1}^{N=3} a_{jk} \sin(\theta_j(t) - \theta_k(t)), \qquad i, j = 1, 2, 3, i \neq j \tag{78}$$

After expansion and simplification we get:

$$\dot{\theta}_{12}(t) = \omega_1 - \omega_2 - a_{13} \sin\theta_{13}(t) + a_{23} \sin\theta_{23}(t), \tag{79}$$

$$\dot{\theta}_{23}(t) = \omega_2 - \omega_3 - a_{12} \sin\theta_{12}(t) + a_{31} \sin\theta_{31}(t), \tag{80}$$

$$\dot{\theta}_{31}(t) = \omega_3 - \omega_1 - a_{13} \sin\theta_{13}(t) + a_{23} \sin\theta_{23}(t). \tag{81}$$

Let us compute the equilibrium. For simplicity, suppose $\omega_1 = \omega_2 = \omega_3 = \omega_0$. Also, let $a_{ij} = 1$. Using Definition 3.1, the equilibria $[\theta_{12}^{eq}, \theta_{23}^{eq}, \theta_{31}^{eq}]^T$ must satisfy:

$$0 = -\sin\theta_{31}^{eq} + \sin\theta_{23}^{eq}, \quad (82)$$
$$0 = -\sin\theta_{12}^{eq} + \sin\theta_{31}^{eq}, \quad (83)$$
$$0 = -\sin\theta_{31}^{eq} + \sin\theta_{23}^{eq}. \quad (84)$$

Note there is an infinite number of triplets $[\theta_{12}^{eq}, \theta_{23}^{eq}, \theta_{31}^{eq}]^T$ that satisfy the above equations. A subset of these equilibria include the infinite set $\left\{[\theta_{12}^{eq}, \theta_{23}^{eq}, \theta_{31}^{eq}]^T \in \mathbb{R}^3 | \theta_{12}^{eq} = \theta_{23}^{eq} = \theta_{31}^{eq} = \theta_0, \;\; \theta_0 \in [0, 2\pi]\right\}$.

Next, we linearize the nonlinear dynamics (79)-(81) around the equilibrium point $\theta_{12}^{eq} = \theta_{23}^{eq} = \theta_{31}^{eq} = \theta_0$. We continue to assume $\omega_1 = \omega_2 = \omega_3 = \omega_0$ and $a_{ij} = 1$. Define the perturbation variables $\tilde{\theta}_{12}(t) = \theta_{12}(t) - \theta_0$, $\tilde{\theta}_{23}(t) = \theta_{23}(t) - \theta_0$, $\tilde{\theta}_{31}(t) = \theta_{31}(t) - \theta_0$. The linearization of (79)-(81) around the given equilibrium point is:

$$\dot{\tilde{\theta}}_{12}(t) = -\cos(\theta_0) \cdot \tilde{\theta}_{31}(t) + \cos(\theta_0) \cdot \tilde{\theta}_{23}(t), \quad (85)$$
$$\dot{\tilde{\theta}}_{23}(t) = -\cos(\theta_0) \cdot \tilde{\theta}_{12}(t) + \cos(\theta_0) \cdot \tilde{\theta}_{31}(t), \quad (86)$$
$$\dot{\tilde{\theta}}_{31}(t) = -\cos(\theta_0) \cdot \tilde{\theta}_{23}(t) + \cos(\theta_0) \cdot \tilde{\theta}_{12}(t). \quad (87)$$

We can write these linearized dynamics in LTI form: $\tilde{x}(t) = A\tilde{x}(t)$ where

$$\tilde{x}(t) = \begin{bmatrix} \tilde{\theta}_{12}(t) \\ \tilde{\theta}_{23}(t) \\ \tilde{\theta}_{31}(t) \end{bmatrix}, \qquad A = \cos\theta_0 \cdot \begin{bmatrix} 0 & +1 & -1 \\ -1 & 0 & +1 \\ +1 & -1 & 0 \end{bmatrix} \quad (88)$$

Interestingly, A is a skew-symmetric matrix. That is, it satisfies $-A = A^T$. The eigenvalues of A are $eig(A) = \cos\theta_0 \cdot \{0, 0 \pm 1.7j\}$. These eigenvalues all have zero real parts. Therefore, we conclude that the Kuramoto oscillator is marginally stable around the equilibrium point $\theta_{12}^{eq} = \theta_{23}^{eq} = \theta_{31}^{eq} = \theta_0$.

The marginal stability property derived above is a consequence of assuming equal natural frequencies $\omega_1 = \omega_2 = \omega_3 = \omega_0$ and homogeneous non-zero coupling $a_{ij} = 1$. It turns out different assumptions can yield different stability properties. We informally summarize these results, which can be formally derived as follows:

- **Stability:** $|\omega_i - \omega_j|$ small & coupling large
- **Instability:** $|\omega_i - \omega_j|$ large & coupling small

Kuramoto oscillators can be adapted to model transient dynamics in both AC power transmission networks and microgrids with DC/AC inverters. The ordinary differential equation at each network node is:

$$M_i\ddot{\theta}_i(t) + D_i\dot{\theta}_i(t) = P_i(t) - \sum_j a_{ij}\sin(\theta_i(t) - \theta_j(t)), \quad (89)$$

where M_i is the generator inertia, D_i is the damping coefficient from droop control, $P_i(t)$ is the injected/extracted power, and a_{ij} represents the maximum power transfer between nodes i and j. Interestingly, increased penetration of renewables (for large networks and microgrids) results in reduced synchronous generator inertia M_i across the network and more transient power injections $P_i(t)$. The consequence of renewable penetration is less robust stability properties – a formidable challenge for power system operators. This can be formally analyzed using tools from this chapter.

To learn more about Kuramoto oscillators, and how they can be used to study voltage collapse in power networks, please read [3] and the references therein.

6 Notes

Modeling dynamical systems is an extraordinarily rich topic – a critical area of study for anyone interested in systems and control. Several textbooks provide excellent expositions, including [4] and [5]. Textbook [4] in particular discusses a general theory for modeling dynamic systems based on a concept known as bond graphs. Reference [5] is rich with examples that are worth reading to build intuition. Readers interested in a deeper understanding of linear systems theory may consult [2,6–8]. These textbooks are typical required references for classical first-year introductory control systems courses. Readers interested in nonlinear systems and stability should consult [1] – the classical reference in this area.

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