Optimal charging of a Vehicle-to-Grid aggregator under power supply constraint

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Abstract

This paper develops a modeling and controlling paradigm to optimize the aggregated charging dynamics of a fleet of plug-in electric vehicles (PEV) under the constraint of supplying a predetermined amount of power back to the grid. A key feature of this model is the use of coupled partial differential equations (PDE) to capture the dynamics of the energy levels of the aggregated fleet of vehicles. The optimal control policy obtained is then validated against a Monte Carlo simulator.

I. Introduction

A. Motivation and Background

As electrification of automobile transportation is soaring and the share of renewable energies in the electricity mix is increasing, the smart charge of Plug-in Electric Vehicles (PEV) is becoming increasingly critical. Hundreds of thousands of multi-kWh batteries connect to the grid on a daily basis, and charge in a straight-forward pattern regardless of the demand for electricity on the grid. Not only can these currently sub-optimal charging habits be improved by taking advantage of off peak hours, but aggregated PEV batteries potentially represent a great means of energy regulation for the grid.

Currently, frequency regulation is provided by generators on a day-ahead market at a MW level. Consequently, if PEVs, with battery powers usually in the 10-20 kW range, are to play a role in regulation, it is necessary to combine their power by using an aggregator. Such an aggregator would enable a fleet of vehicles to meet the minimum threshold to enter the market while at the same time enabling a more efficient charge of vehicles.

Load control of PEVs can be challenging because of uncertainties and specificities associated with each driver. In this work, we consider a PEV aggregator, which stocks, charges PEVs and dispatch them to customers. Optimal PEV charging schedule has often addressed to solve the single-vehicle problem; however, for the purpose of interacting with the grid, an aggregator needs to deal with many EVs and needs a highly scalable optimization tool. In this work, we design a scalable optimization framework to optimize the charging schedule of a Vehicle-To-Grid fleet.
Section II will develop a transport-based method to model the aggregate charging dynamics of a Vehicle-To-Grid fleet; Section III will develop a Linear Program to solve the optimal charging problem.

B. Relevant literature

In the recent literature, a number of modeling frameworks and optimization methods have been used to deal with the problem of optimal charging patterns of PEVs. Most of the studies have considered the problem for a single vehicle.

S. Bashash develops in [1] a Qradratic Programming framework to find the optimal cost charging pattern of one vehicle with varying electricity prices. This method gives a computationally efficient way to solve the problem for a single vehicle.

Soohe Han [2] has faced the problem from an aggregation point of view and has developed a dynamic programming solver to maximize the revenue of the aggregator. The model framework is very close from the individual vehicle point of view in term of Cost formulation and solving methods; it gives interesting insight into the aggregation problem and proposes a

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solution based on 'utility weights', which depend on the state of charge (SOC) of vehicles, to adapt charging patterns to vehicles. However, the latter model considers that the aggregator is allowed to supply any amount of power to the regulation and that the user drive-cycles are deterministic.

Bashash and Fathy in [3] have developed a first-order bilinear transport PDE, representing the charging dynamics a G2V fleet; the authors developed an optimal control algorithm for the use of "intelligent" charging stations, which can adapt the charging power that is uniformly provided to plugged-in vehicles; therefore charging pattern of the fleet is unique and the optimal control is solved for a unique variable.

C. Focus of this study

The main contribution of this paper is to model the optimization problem of charging an aggregated fleet of vehicles under V2G constraints, by a system of coupled partial differential equations. These equations capture the dynamics of the aggregated energy level of the PEV batteries. We seek to find the optimal charge and discharge policy in order to reach the lowest cost of charge while supplying a predetermined amount of power to the grid.

In this study we assume the price of electricity is known prior to the time interval. This assumption is relevant since in reality prices are known 24 hours ahead of time. We also consider deterministic driving cycles. In other words, the time when drivers leave with their vehicle and return is known in advance.

D. Nomenclature

In the following work, we will use the nomenclature:

w	Distribution of charging car
v	Distribution of idle cars
γ	Distribution of discharging cars
$\sigma_{i\to c}$	Flow of vehicles from Idle to Charge
$\sigma_{i\to d}$	Flow of vehicles from Idle to Discharge
q	Instantaneous charging power
X_{max}	Storage capacity of batteries
X_{dep}	Min allowed charge for cars to depart
X_{Dmin}	Minimum allowed charged for discharging and idle cars
X_{Cmax}	Max allowed charge for charging cars
N_{min}	Min number of cars required to be charged enough at T_{max}
C_{elec}	Price of electricity
P_{grid}^{des}	Power required by the grid

II. Continuous V2G model development

A. Continuous Transport-Based model

This section details the use of a hyperbolic partial differential equation (PDE) to model the dynamics of the aggregated charge of the PEVs. We assume that all the vehicles that are charging do so at a constant rate q_c and that the operator can control which of the plugged vehicles are charging. The total power demand for charging these PEVs is proportional to the number of vehicles that are actually charging.

The charging PEVs can be considered as dynamic loads diffusing towards higher SOC at the charging rate q_c.

The distribution of the number of cars that are charging at their docking station in function of the state of charge is approximated by a continuous function, as represented in figure 1.

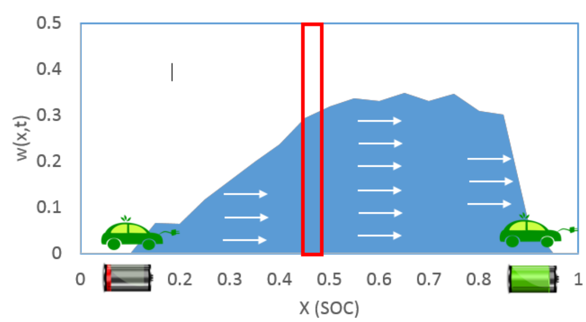

Figure 1: Distribution of the number of vehicles for a given state of charge. Since PEVs are charging this curve diffuses towards higher SOCs

To describe with a mathematical expression the transport process from low SOC to higher SOC, we consider the number of vehicles charging at time t with a state of charge x: $w(x,t)$.
These vehicles are charging at a speed q_c which corresponds to the transport rate.
We then consider an infinitesimal slice of vehicles whose SOC is comprised between x and $x+dx$. The variation of the number of vehicles in that SOC range between time t and $t+dt$ corresponds to the difference between the entering flux and the exiting flux, respectively $w(x-q_c dt,t)dt$ and $w(x,t)dt$. The variation of loads inside the control volume also depends on the flux of vehicles that plug into the grid at a moment

between t and $t+dt$. This source term corresponding to the algebraic number of vehicles that start charging at time t with a SOC of x is noted $\sigma_{i\to c}(x,t)$.

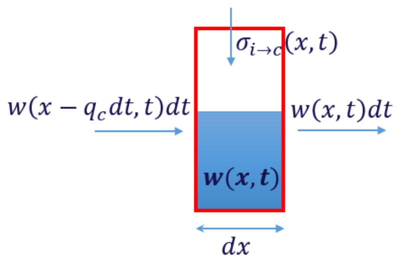

Figure 2: Algebraic flows in the control volume of charging vehicles

Consequently, a conservation law applied to the number of charging vehicles yields

$$w(x,t+dt)dx = w(x,t)dx + w(x-q_c dt,t)dt - w(x,t)dt$$
$$+\sigma_{i\to c}(x,t)dxdt$$

This equation can be rewritten as

$$\frac{\partial w}{\partial t}(x,t) = -q_c\frac{\partial w}{\partial x}(x,t) + \sigma_{i\to c}(x,t) \tag{1}$$

Equation (1) is the governing PDE of the dynamics of the distribution of charging cars.

To define the boundary conditions of (1) we assume all the vehicles have a SOC greater then X_{min} and lesser than X_{max}. In other words, the number and flux of cars for $x \leq X_{min}$ is null.

$$w(x,t) = 0 \quad \forall x \leq X_{min}$$

B. Dynamics of a V2G fleet

Three-state model To complete the derivation of our continuum level charging dynamics of a V2G fleet, we must consider three possible states for the vehicles. PEVs can either be

- Charging, retrieving power from the grid and increasing SOC.
- Discharging, supplying power back to the grid (V2G mode).
- Idle, plugged into the grid but with no interaction.

There is also a fourth distinct state, which the aggregator does not control:

- Vehicles on the road or not connected to their charging station

In this model, the aggregator has control only over vehicles that are plugged into their charging station and not over the vehicles that are on the road. The control policy therefore applies transfers between the first three states. The global dynamics are mathematically described by coupled PDEs.

Global dynamics The equation detailed in part A models the dynamics of charging vehicles. Similarly, the dynamics for the vehicles that are discharging (providing power to the grid) are modeled by a transport equation. For discharging cars, the transport rate is $-q_d$: in this case the distribution of vehicles diffuses towards the left from high SOCs to lower SOCs as vehicles use battery energy to provide power to the grid.

$$\frac{\partial \gamma}{\partial t}(x,t) = q_d\frac{\partial \gamma}{\partial x}(x,t) + \sigma_{i\to d}(x,t) \tag{2}$$

Finally, the number of idle vehicles $v(x,t)$ corresponds to the remaining vehicles, which are plugged into their charging station, but are neither charging nor discharging. Here we must also consider vehicles coming from a third source term $\sigma_{i\to or}(x,t) = Dep(x,t) - Arr(x,t)$. This represents the algebraic number of vehicles plugging into their charging station: a negative number means more vehicles are leaving the garage than vehicles are returning. This input is uncontrollable since we can not decide when drivers start using their vehicles or stop using them. The variation of the number of idle vehicles is therefore the difference between the three transfer terms.

$$v_t(x,t) = (Arr(x,t) - Dep(x,t)) - \sigma_{i\to c}(x,t) - \sigma_{i\to d}(x,t)$$

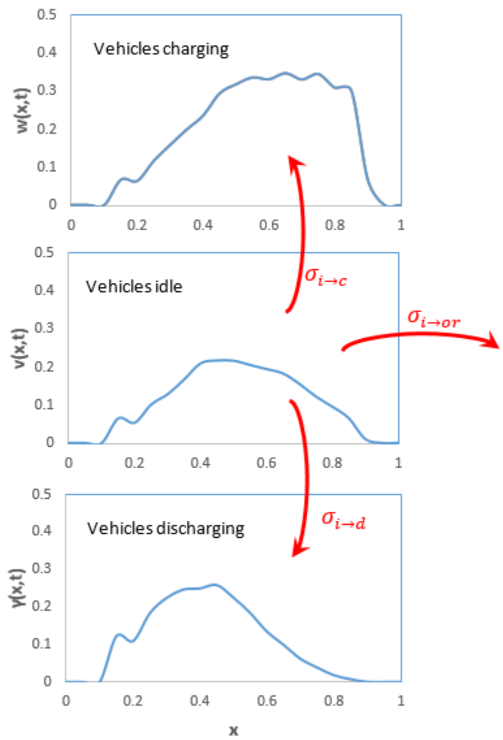

Figure 3: Conservation of the number of vehicles and transfers between different states

Figure 3 represents the three possible states for the plugged vehicles as well as the three transfers between them. At this stage it is important to note that the control variables are $\sigma_{i\to c}$ and $\sigma_{i\to d}$, the transfer between states w,γ and v. and that, $\sigma_{i\to or}(x,t)$ is taken as an uncontrollable input.

The system of coupled PDEs mathematically represents the dynamics of the aggregated load of charging, discharging and idle vehicles.

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) = -q_c \frac{\partial w}{\partial x}(x,t) + \sigma_{i\to c}(x,t) \\ \\ \frac{\partial \gamma}{\partial t}(x,t) = q_d \frac{\partial \gamma}{\partial x}(x,t) + \sigma_{i\to d}(x,t) \\ \\ \frac{\partial v}{\partial t}(x,t) = -\sigma_{i\to or}(x,t) - \sigma_{i\to c}(x,t) - \sigma_{i\to d}(x,t) \end{cases}$$

The appropriate boundary conditions and initial conditions are developed in the next section.

III. Optimal control of loads

A. System modeling and constraints

The objective here is to design the best distribution of vehicles between charging, discharging and idle states over time. The aggregator minimizes the cost of charging vehicles over the time period T_{max}. At time t, the aggregator only charges vehicles that are in the charging state $W(t) = \int_0^{X_{max}} w(x,t)dx$. Then the total cost over time is given by:

$$C = \int_{t=0}^{T_{max}} C_{elec}(t) \int_0^{X_{max}} w(x,t)qdxdt \qquad (3)$$

To ensure physical meaning of the system, we impose boundaries on SOC values for each category: for $x > X_{Cmax}$, cars cannot charge any more and for $x < X_{Dmin}$, cars are forced to charge:

$$\begin{aligned} w(x,t) &= 0 \ \forall x \geq X_{Cmax} \\ \gamma(x,t) &= 0 \ \forall x \leq X_{Dmin} \\ v(x,t) &= 0 \ \forall x \leq X_{Dmin} \end{aligned} \qquad (4)$$

The system is faced with two different types of constraints: following the power demand from the grid and meeting the demand of vehicles from customers.

Power supply constraint We consider that the V2G aggregator sells electricity in a day ahead market: the amount of power $P_{grid}^{des}(t)$ is known one day in advance. This condition sets the number of discharging cars $\Gamma(t) = \int_0^{X_{max}} \gamma(x,t)dx$ over time:

$$P_{supply}(t) = \int_0^{X_{max}} \gamma(x,t)qdx = P_{grid}^{des}(t) \ \forall t \qquad (5)$$

Drivers' demand constraint We consider that the demand and arrivals of cars are known one day in advance. This could be enabled by a reservation-based system; in practice these functions can represent expected arrivals and demands. The arrival of cars $Arr(x,t)$ is known for all time and all SOC values. Similarly the total demand of vehicles over time $Dem(t)$ is known one day in advance. It remains to the aggregator system to optimize the distribution of leaving cars $Dep(x,t)$ over SOC values. We formulate this problem by fixing a minimum required charge level X_{dep} to allow a car to be taken by a driver. Then, $Dep(x,t)$ becomes a controllable input, which satisfies the following constraint:

$$\int_{X_{dep}}^{X^{max}} Dep(x,t)dx = Dem(t) \ \forall t \tag{6}$$

Time horizon and final condition The problem here is a finite-horizon optimization: minimizing cost over time period T_{max}. To ensure continuity of the system after time period T_{max}, we impose that the system contains a minimum number of vehicles N_{min} that are able to depart after T_{max}.

$$\int\limits_{X_{dep}}^{X^{max}} (w+\gamma+v)(x,T_{max})dx \geq N_{min} \tag{7}$$

B. Formulation of the optimization problem

Eventually, we derive the resulting optimization problem:

$$\min_{\sigma_{i\to d},\sigma_{i\to c},Dep} C = \int\limits_{t=0}^{T_{max}} C_{elec}(t) \int\limits_{0}^{x_{max}} w(x,t)qdxdt$$

subject to

$$\begin{aligned} & w_t(x,t) = -q_c w_x(x,t) + \sigma_{i\to c}(x,t) \\ & \gamma_t(x,t) = q_d \gamma_x(x,t) + \sigma_{i\to d}(x,t) \\ & v_t(x,t) = -[\sigma_{i\to c}(x,t) + \sigma_{i\to d}(x,t)] \\ & \qquad + Arr(x,t) - Dep(x,t) \end{aligned}$$

$$\begin{aligned} & \sigma_{i\to d}(x,t) \in [-\gamma(x,t), v(x,t)] \\ & \sigma_{i\to c}(x,t) \in [-w(x,t), v(x,t)] \end{aligned}$$

$$\begin{aligned} & w(.,0) = w_0, \gamma(.,0) = \gamma_0, v(.,0) = v_0 \\ & (4), (5), (6), (7) \end{aligned}$$

We can remark here that all the functions and constraints are linear regarding the variables w, v and γ. This is a key property of the model, which allows use of Linear Programing.

Discretization We note $n \in [0, N]$ the index for discretization in time with time step Δt, $T_{max} = N\Delta t$. We note $j \in [0, J]$ the discretization index for State of Charge with step Δx. The function variables are discretized into vectors: $f_j^n = f(j\Delta x, n\Delta t)$.

We choose a Lax-Wendroff numerical scheme to discretize the PDEs. This numerical scheme has good properties when applied to hyperbolic PDEs. The main condition of this numerical scheme is the Courant-Friedrichs-Lewy condition $q\frac{\Delta x}{\Delta t} \leq 1$ where q is the celerity. Appendices A and B give more details about discretization choices.
Boundary conditions are necessary to ensure that the PDE problem is well formulated. For transport PDEs, the necessary point depends on the sign of the speed propagation. For charging cars, we need a boundary condition in $x = 0$, for discharging cars we need a boundary condition in $x = X_{max}$. For charging cars, the boundary condition is the absence of a source term in $x = 0^-$: this means $w_0^n = 0$; we have the same boundary condition in $x = X_{max}$ for discharging cars.

Linear Program We note M_c and M_d the transition matrices derived from the discretization of PDEs (see Appendix A). The dynamics of charging and discharging cars reads

$$\begin{aligned} w^{n+1} &= M_c w^n + \sigma_c^{n+1} \\ \gamma^{n+1} &= M_d \gamma^n + \sigma_d^{n+1} \end{aligned}$$

After discretization of all the functions, we solve the optimization problem for the optimal values of w, v, γ and Dep.
The coupled PDE constraints result in the transition constraint:
$[w+\gamma+v]^{n+1} + \frac{Dep^{n+1}}{\Delta x} = M_c w^n + M_d \gamma^n + \frac{Arr^{n+1}}{\Delta x}$

Then, the final Linear Problem reads:

$$\min_{w,v,\gamma,Dep} \Delta t \Delta x q \sum \sum C_{elec}^n w_j^n$$

subject to

$$[w+\gamma+v]^{n+1} + \frac{Dep^{n+1}}{\Delta x} = M_c w^n + M_d \gamma^n + \frac{Arr^{n+1}}{\Delta x}$$
$$w_j^n = 0, \forall j > Jc$$
$$\gamma_j^n = 0, \forall j < Jd$$
$$v_j^n = 0, \forall j < Jd$$
$$w^0 = w_{init},\ \gamma^0 = \gamma_{init},\ v^0 = v_{init}$$
$$w, v, \gamma, Dep \geq 0$$

$$q\Delta x \sum_j \gamma_j^n \geq P^n$$

$$\sum_{j=J_{dep}}^{J_{max}} Dep_j^n = Dem^n$$

$$\Delta x \sum_{j=J_{dep}}^{J_{max}} w_j^N + v_j^N + \gamma_j^N \geq N_{final}^{min}$$

IV. Results and validation

In this section, we first examine the performance of the model to control the charging schedule of a fleet of 500 vehicles. Then, we derive discrete integer controls and validate the result with a Monte Carlo simulator.

A. Case study

The initial fleet contains 500 vehicles; the arrivals and departures of cars are shown figure 4. The total number of cars which are plugged-in vary from 213 to 525. This distribution is chosen to fit with a real case scenario where many cars leave during the morning and come back during evening or night.

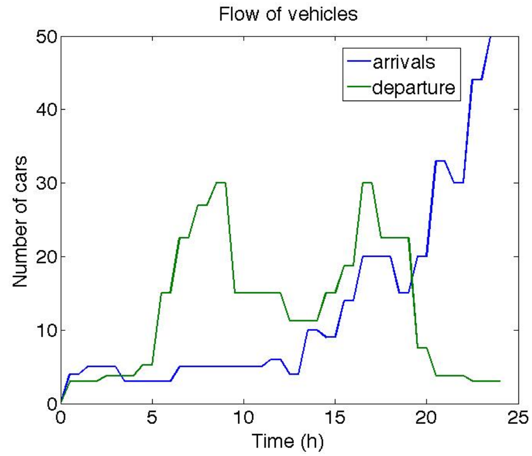

Figure 4: Arrivals and departure of cars

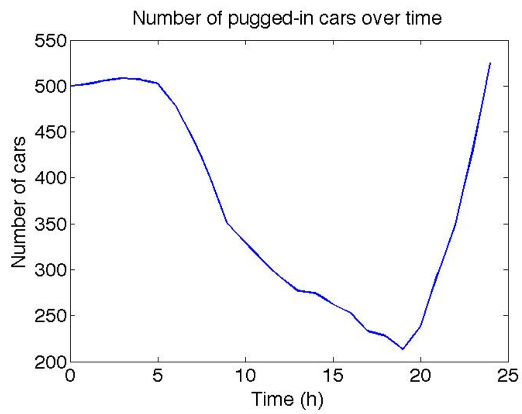

Figure 5: Number of cars in the system

The price of electricity and power required by the grid are shown figure 6 and 7. The parameters of the battery are chosen as follows:

- Instantaneous charging power: q=2kW
- Battery storage capacity : $X_{max} = 38$kWh
- $X_{dep} = 28.5$ kWh
- $X_{Cmax} = 36.1$ kWh, $X_{Dmin} = 7.6$kWh
- $N_{min} = 80$

B. Control implementation

The discretization steps are $\Delta t = 30$min and $\Delta x = 1$kWh. We apply the Linear Program to the system to get the best distribution of cars over time and State of Charge.
The total number of cars in each category (idle, discharge and charge) is shown figure 6.

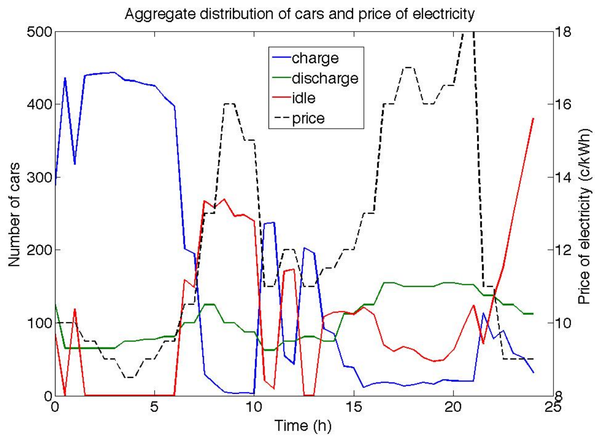

Figure 6: Optimal distribution of cars over time and Price of electricity

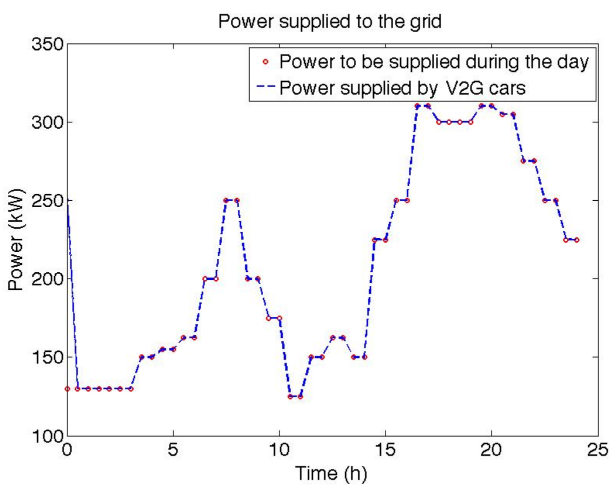

Figure 7: Power required by the grid and power generated by V2G

The controller is able to force enough cars in the discharging category to meet the power supply demand (figure 7); then the remaining plugged-in cars are managed between charge and idle in order to meet demand from drivers and to minimize the overall cost. Figure 6 shows flows from charge to idle during peak hours; in this case study, cars mainly charge from 1 am to 6 am, when a lot of cars are still plugged-in and the price of electricity is low.

A second aspect of the controller is the optimization of the distribution of cars along SOC values. Figure 8 and 9 show how the controller tends to aggregate cars around the typical SOC values of the system. At the end of the optimization period, cars are mainly in the idle category and are charged at the minimum required value $X_{Dmin} = 7.6$kWh, a second peak of cars is aggregated at $X_{dep} = 28.5$kWh, which corresponds to the final constraint of having a minimum number of cars $N_{min} = 80$ charged for the second day. The controller also optimizes at which SOC level cars depart with respect to the condition $x \geq X_{dep}$. Figure 9 shows that almost all cars depart with the minimum required charge value.

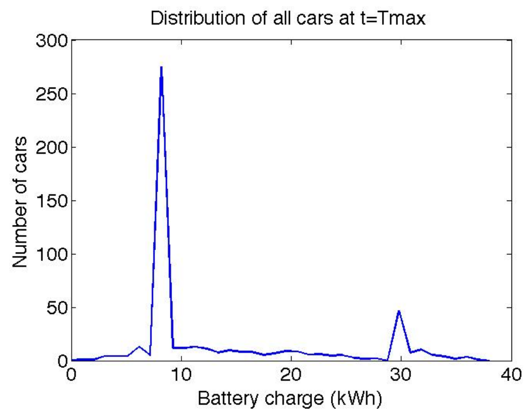

Figure 8: Distribution of cars along SOC values at Tmax

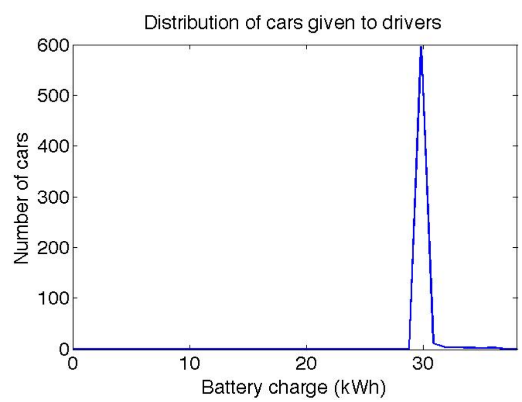

Figure 9: Distribution of cars that departed along SOC values

C. Discrete control and validation

In this section we explain how the control can be applied to real fleets of vehicles. The continuous model gives an optimal continuous control over SOC distributions. However, in a real case scenario, only dis-

crete values of SOC and rounded number of cars can be controlled.
We validate our model using a Monte-Carlo-style simulation: the control obtained form the LP formulation is transformed in rounded flows of cars. The initial fleet is modeled with a discrete approach : each car is modeled one by one and can be assigned to 4 different states : charging, discharging, idle or absent. For each of these states, the dynamic of the battery charge is modeled as following: if the state is 'Charge', the battery charges at rate q; if the state is 'Discharge', the battery discharges at rate q; if the state is 'Idle', the charge remains constant; if the state is 'Absent', the battery charge is not tracked. The discrete control is applied to this fleet of vehicles and the evolution of the system is simulated during N time steps, which correspond to the same $T_{max} = 24h$. Figure 10 summarizes the steps of the simulation.

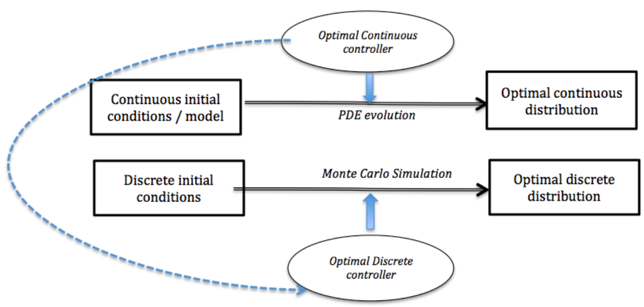

Figure 10: Global scheme of controller

The distributions of cars resulting from the discrete controller are presented figure 11 and 12. The graphs show that the passage of the controller from continuous to discrete cases has very satisfying results. The resulting distributions are very close when we look at the aggregated level.

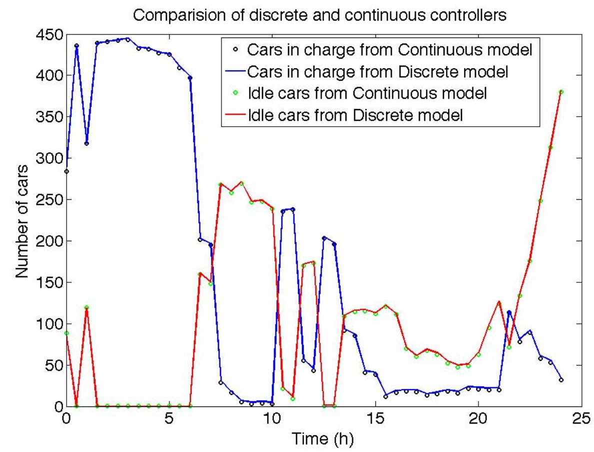

Figure 11: Distribution of cars along SOC values at Tmax

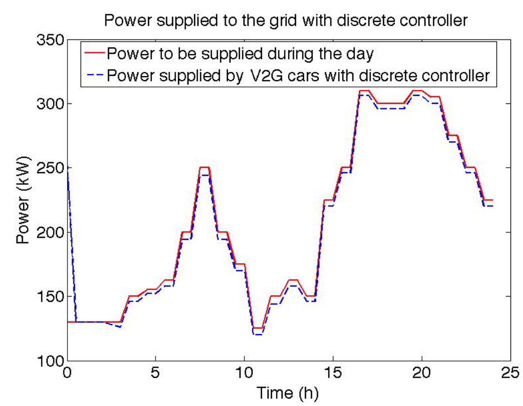

Figure 12: Power supplied by V2G with discrete control

However, for the purpose of examining the suitability of the continuous model, we have taken very similar evolutions of charge between the Monte Carlo simulator and the continuous simulator. To get more insight into the robustness of the model, one could add more complexity to the Monte Carlo simulator and test the resulting aggregation controller.

D. Smoothing of controls

The impact of frequent controls on battery degradation remains unclear. The continuous model allows to shift the global control policy towards smoother and less frequent controls. For this purpose, we can add to the cost function a penalization for high values of controls. We derive the following penalization:

$$\begin{aligned} Pen &= \lambda(||(\sigma^n_{i\to d})_n||_1 + ||(\sigma^n_{i\to c})_n||_1) \\ &= \lambda\Big(||(w^{n+1} - M_c w^n)_n||_1 \\ &\quad + ||(\gamma^{n+1} - M_d \gamma^n)_n||_1\Big) \end{aligned}$$

This is very similar to adding a L_1 penalization on the derivative of the state function.
It is worthy to note that $||(w^{n+1} - M_c w^n)_n||_1 + ||(\gamma^{n+1} - M_d\gamma^n)_n||_1$ is a convex function with respect to the variables (w, γ, v, Dep). Hence, we use the optimization solver CVX [7] to solve the problem with the penalized cost. The resulting optimal distribution of cars is shown figure 13

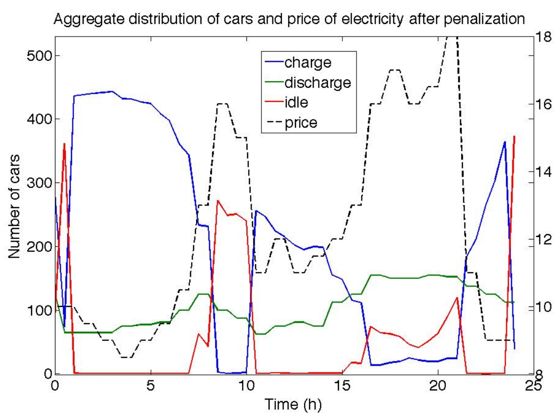

Figure 13: Aggregate distribution of cars after penalization L1 with $\lambda = \frac{\max(C_{elec})}{1.4}$

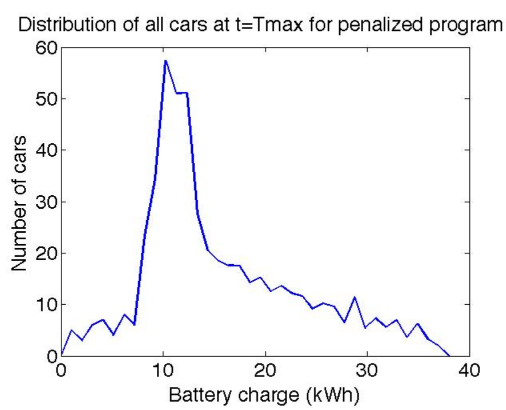

Figure 14: Final distribution of cars along SOC values after penalization L1

Figure 13 shows that the penalized optimal problems drives to more 'global' policies. As in the previous case, the controller forces the number of discharging cars to meet the power demand; then the fleet tends to be either completely idle during peak hours, or completely charging when prices of electricity are low. Figure 14 shows that the cars are more equally distributed across SOC values, even though X_{minD} is still an aggregation point.

V. Conclusion and Discussion

In this report we developed a modeling and control program to optimize the charging and discharging schedule of a fleet of PEVs, which supply energy to the grid on a day ahead market. We modeled a system where vehicles can be assigned to one of the three states: charging, discharging and idle. We formulated the collective charging dynamics of each category using transport-based modeling principles, and coupled the resulting PDEs to model the global evolution of the system. We examined different finite difference methods to discretize the system of PDEs and get a stable numerical scheme. We developed a Linear Program to solve the charge and discharge schedule in the discretized spaces. The optimization gives at each time step the optimal flows of cars along SOC values; results show that the optimization program can both meet grid and drivers' requirements and control for the cost of charging vehicles. Controlling for both time and SOC distributions gives large freedom to the controller. From the LP results, we derived discrete, integer controls to be applied the fleet of vehicles. We modeled the fleet of vehicles with a Monte Carlo simulator which takes the same assumptions for parameters than the continuous model. Results show that the performance of the integer controller is very close to the continuous'.

This work has developed an innovative framework to model and control fleets of vehicles. This framework allows to solve optimal charging and dispatching problems in both time and SOC spaces in a highly scalable way. The technique mainly relies on Transport-Based models with transfers between different states, which represent charge, discharge and idle. This framework is very suitable to describe the dynamics of most loads. Hence, this modeling technique is very promising to address future optimization challenges of dealing with demand-Side management and high numbers of loads.

A. Appendix I. Discretization

In this section we describe the discretization method applied to the PDEs of the form

$$w_t(x,t) = -q_c w_x(x,t) + \sigma_{i\to c}(x,t).$$

Two particularly interesting characteristics of the Lax-Wendroff schemes for a transport equation are

- Second order accuracy in time and space achieved with a relatively low computational intensity.
- Stability for transport equations under the CFL condition.

The numerical method based on explicit time integration is detailed below. We note $n \in [0,N]$ the index for discretization in time with time step Δt,and $j \in [0,J]$ the discretization index for State of Charge with step Δx. Define

$$w_j^n = w(x_j, t_n) = w(j\Delta x, n\Delta t)$$

with the Taylor series

$$w(x,t+\Delta t) = w(x,t) + \Delta t w_t(x,t) + \frac{\Delta t^2}{2} w_{tt} + O(\Delta t^3)$$

Since

$$w_t(x,t) = -q_c w_x(x,t) + \sigma_{i\to c}(x,t)$$

which gives

$$w(x,t+\Delta t) = w(x,t) - q_c \Delta t w_x + \frac{q_c^2}{2} w_{xx} + \Delta t \sigma_{i\to c}$$

$$-\frac{q_c \Delta t^2}{2}(\sigma_{i\to c})_x + \frac{\Delta t^2}{2}(\sigma_{i\to c})_t$$

Replacing the derivatives by their approximate

$$\begin{aligned} w_x(x,t) &\approx \frac{w(x_j+\Delta x,t) - w(x_j,t)}{\Delta x} \\ &\approx \frac{w_{j+1}^n - w_j^n}{\Delta x} \end{aligned}$$

We obtain

$$w_j^{n+1} = M_c^*[w_{j-1}^n, w_j^n, w_{j+1}^n]^T + \sigma_{i\to c})_j^{n+1}$$

Where vector M_c^* is defined as

$$M_c^* = \begin{pmatrix} \frac{C}{2} + \frac{C^2}{2} & 1 - \frac{C^2}{2} & -\frac{C}{2} + \frac{C^2}{2} \end{pmatrix}$$

Matrix M_c can then be written

$$M_c = \begin{bmatrix} M_c^0 & 0 & \cdots & \cdots & 0 \\ 0 & M_c^* & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & M_c^* & 0 \\ 0 & \cdots & \cdots & 0 & M_c^K \end{bmatrix}$$

The parameter $C = q_c \frac{\Delta x}{\Delta t}$ often called the Courant number is particularly important. Indeed, the scheme is stable if $C \leq 1$, which creates a constraint on the choice of the time steps and space steps. It is important to note however that the Courant Friedrich Lewy condition is not a sufficient condition for the approximate solution to converge to the analytic solution.

B. Appendix II. Discretization accuracy

A. Analytical solution

In order to test the validity of our numerical method, we solve the wave equation analytically in some simple cases and confront them to the numerical results.

The analytical solution is derived by using the method of characteristics. In the case of a simple hyperbolic equation the characteristics are simply lines with a slope corresponding to the velocity q_c written as c in this section. Depending on where the characteristics originate from, we either choose the boundary condition or the initial condition. A simple case where the input is constant over time is presented below.

Initial condition: $w(x,t=0) = w_0(x) \;\; \forall x$

Boundary condition: $w(x=0,t) = 0 \;\; \forall t$

Source term: $\sigma_{i\to c}(x,t) = \alpha + \beta x \;\; \forall x,t$

As a result

$$\begin{cases} w(x \geq ct, t) = w_0(x-ct) + \alpha t + \frac{\beta}{2c}(2xct - c^2t^2) \\ w(x \leq ct, t) = \frac{\alpha x}{c} + \frac{\beta x^2}{2c} \end{cases}$$

B. Comparison between numeric and analytic results

We illustrate here the importance of the choice of the Courant number which influences significantly the accuracy of the result.

Because the source term is not smooth and in most cases contains important discontinuities, if $q_c \frac{\Delta x}{\Delta t} < 1$, artifacts similar to the Gibbs phenomenon with unwanted oscillations, are created. Figure 15 illustrates this for $C = 0.5$.

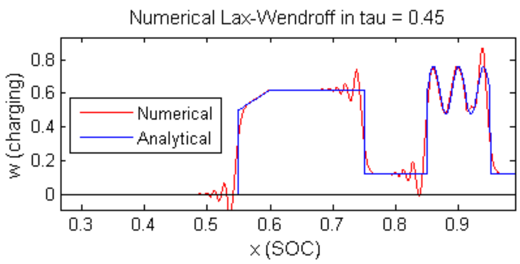

Figure 15: Example of numerical solving of the wave equation with $C = 0.5$

However, if the time step is chosen more wisely in order to have $C = 1$ the match between numeric results and the real solution is largely improved and the relative error is at maximum of the order of 10^{-3}.

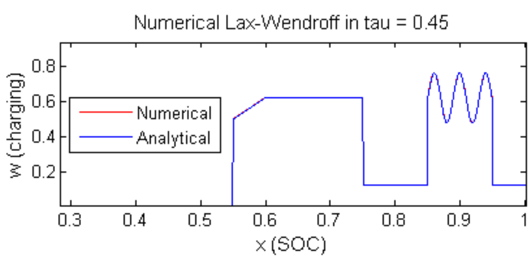

Figure 16: Example of numerical solving of the wave equation with $C = 1$

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