# Robustness of Boundary Observers for Radial Diffusion Equations to Parameter Uncertainty

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*Abstract*— **Boundary observers for radial diffusion equations can be derived to achieve exponential convergence of the estimation error system provided that coefficients are known; which can be either constant or possibly spatially and time varying. When the coefficients depend on the state, their values are not longer known and this might prevent the estimation error to converge to zero. Here, we address the state estimation problem for a radial diffusion equation in which the diffusion coefficient depends on the spatial average of the state value; using an observer with a constant diffusion coefficient. The error introduced to the observer, in this particular situation, can be quantified from an input-to-state stability (ISS) analysis. This study is motivated mainly by the problem of state estimation from electrochemical models of lithium-ion batteries, namely the Single Particle Model (SPM). In this application, the variation in the diffusion coefficient as a function of the spatial average of the states is of several orders of magnitude. We consider this result an additional effort in the broader goal of designing estimation algorithms from electrochemical models of lithium-ion batteries without relying in the discretization of the PDEs in these models.**

## I. INTRODUCTION

### A. Motivation

Lithium-ion technology is a common choice among the rechargeable battery family due to several attractive characteristics: high power and energy storage density, lack of memory effect and low self-discharge [1]. It has a wide employment in portable electronics and an increasing interest for electrified transportation [2] and grid energy storage.

The safe and optimal use of lithium-ion batteries relies on accurate estimation of electrochemical states and parameters [3]. The availability of detailed electrochemical models [4] is driving a recent effort to design of model-based estimation algorithms; however, the complexity of these models also pose various challenges. One aspect of the models that adds complexity is the dependence of some parameters on the states. The rate at which lithium ions diffuse through the porous electrodes in the battery is one of such parameters. For example, in [5], it was noted that the diffusion coefficient of an NMC electrode varies almost three orders of magnitude as a function of the state of charge.

### B. Contribution

We derive an observer for a radial diffusion equation in $n$-dimensional balls with boundary measurements. When coefficients are constant and known, the observer provides exponentially convergence with an arbitrary convergence rate. On the other hand, if the diffusion coefficient is a function of the state, in particular of the spatial average of the state, an error arises in the estimation due to the fact the value of the diffusion coefficient is not longer known. The main contribution of this paper is the derivation of bounds in the estimation error that arises in this particular situation. The derivation of these bounds follow recent results on the input-to-state stability of one-dimensional parabolic equations [6], [7]. The main technical challenge is to verify the results in [6], [7] are also valid for radial diffusion equations in $n$-dimensional balls.

The observer design follows the PDE backstepping method. This method has been used for the stabilization of various unstable PDE systems [8]. In [8], backstepping boundary controllers and observers are designed for some unstable parabolic, hyperbolic PDEs and other types of PDEs. Boundary stabilization and estimation of diffusion-reaction equations in $n$-dimensional balls was introduced in [9], [10]; the extension to spatially varying coefficients was derived for the case $n = 2$ in [12] and for the case $n = 3$ in [11]. Boundary observers have been derived previously for simplified electrochemical models of lithium ion batteries, namely, the Single Particle Model (SPM): state and parameter estimation was studied in [13] and [14], state estimation for cells with multiple active materials in [15], an observer for the SPM with electrolyte dynamics was derived in [16] and one for the SPM with averaged thermal dynamics in [17]. In all of these cases, the diffusion coefficients appearing in the SPM model were assumed to be known and independent of states.

We consider the result presented in this paper as an additional step in the broader effort to design estimation and control algorithms for lithium-ion batteries from electrochemical models without relying on the discretization of the PDEs in these models.

### C. Organization

The rest of this paper is organized as follows. The problem statement is presented in Section II. The main result appears in Section III. The single particle model is briefly described in Section IV and the corresponding observer is derived in V. Final remarks appear in Section VI. The proof that conditions and assumptions in [6], [7] are satisfied for radial diffusion equations are included in the Appendix.

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## II. PROBLEM STATEMENT

### A. *Diffusion with Average-Value-Dependent Coefficients*

Consider the radial diffusion equation

$$u_t(x,t) = \frac{\epsilon\left(\overline{u}(t)\right)}{x^{n-1}}\left[x^{n-1}u_x(x,t)\right]_x, \tag{1}$$

for $x \in (0,1), t > 0$, $n \in \mathbb{N}$, with boundary conditions

$$u_x(0,t) = 0, \tag{2}$$
$$\epsilon(\overline{u}(t))u_x(1,t) = f(t), \tag{3}$$

and initial condition $u_0(x) \in \mathcal{C}^2\left([0,1]\right)$ and some $f(t) \in C^2((0,\infty))$ given. The diffusion coefficient $\epsilon : [u_{\min}, u_{\max}] \to (0,\infty)$ in (1) is an *affine* function of the spatial average value $\overline{u}(t)$, defined as

$$\overline{u}(t) = n\int_0^1 u(x,t)x^{n-1}dx. \tag{4}$$

Equation (1) with boundary conditions (2)-(3) describe the *radial* diffusion of the quantity $u$ in a $n$-dimensional sphere. The boundary value $y(t) = u(1,t)$ is known and the goal is to find an estimate of $u(x,t)$ from the boundary measurements $f(t)$ and $y(t)$.

### B. *Observer Design*

The proposed observer is copy of the plant (1)-(3) with linear output error injection, that is

$$\widehat{u}_t(x,t) = \frac{\epsilon(\overline{u}_\star)}{x^{n-1}}\left[x^{n-1}\widehat{u}_x(x,t)\right]_x + P(x)\widetilde{u}(1,t), \tag{5}$$

for $x \in (0,1)\,, t > 0$ with boundary conditions

$$\widehat{u}_x(0,t) = 0, \tag{6}$$
$$\epsilon(\overline{u}_\star)\widehat{u}_x(1,t) = f(t) + Q\widetilde{u}(1,t), \tag{7}$$

initial conditions are $\widehat{u}_0(x) \in \mathcal{C}^2\left([0,1]\right)$ and a *fix* value $\overline{u}_\star \in [u_{\min}, u_{\max}]$ used to compute a constant diffusion coefficient. In (5) and (7), $P(x)$ and $Q$ are in-domain and boundary observer gains, respectively. Since $\epsilon$ is affine, the difference between the diffusion coefficient in the plant and the diffusion coefficient in the observer is proportional to the error between $\overline{u}(t)$ and $\overline{u}_\star$, that is, for some $\epsilon_1 \in \mathbb{R}$ and $\delta\overline{u}(t) = \overline{u}(t) - \overline{u}_\star$, it follows that

$$\epsilon(\overline{u}(t)) - \epsilon(\overline{u}_\star) = \epsilon_1\delta\overline{u}(t). \tag{8}$$

The estimation error is defined as $\widetilde{u}(x,t) := \widehat{u}(x,t) - u(x,t)$, and the estimation error system is obtained by subtracting (5)-(7) from (1)-(3), that is

$$\begin{aligned}\widetilde{u}_t(r,t) =& \frac{\epsilon\left(\overline{u}_\star\right)}{x^{n-1}}\left[x^{n-1}\widetilde{u}_x(x,t)\right]_x - P(x)\widetilde{u}(1,t) \\ &+ \delta\overline{u}(t)\frac{\epsilon_1}{x^{n-1}}\left[x^{n-1}u_x(r,t)\right]_x,\end{aligned} \tag{9}$$

for $x \in (0,1)$, $t > 0$, with boundary conditions

$$\widetilde{u}_x(0,t) = 0, \tag{10}$$
$$\epsilon\left(\overline{u}_\star\right)\widetilde{u}_x(1,t) = -\epsilon_1\delta\overline{u}(t)u_x(1,t) - Q\widetilde{u}(1,t), \tag{11}$$

and initial conditions $\widetilde{u}_0(x) = u_0(x) - \widehat{u}_0(x), \widetilde{u}_0(x) \in C^2\left([0,1]\right)$. The estimation problem is now the problem of choosing gains $P(x)$ and $Q$ to guarantee some stability properties of the estimation error system. More precisely, we will choose $P(x)$ and $Q$ such that $\|\widetilde{u}(x,t)\|_2$ is bounded by a term that is asymptotically proportional to $\epsilon_1$. Thus, when $\epsilon_1 = 0$, this choice of $P(x)$ and $Q$, will imply that $\|\widetilde{u}(x,t)\|_2 \to 0$ as $t \to \infty$.

## III. STABILITY OF THE ESTIMATION ERROR SYSTEM

Before showing the stability properties of the estimation error system in (9)-(11), a set of constants need to be defined.

*Definition 1:* The positive scalars $A_2$ and $A_3$ quantify the effect of a discrepancy in the value of the diffusion coefficient on the value of the estimation error. These two quantities are computed from the plant and observer parameters as follows

$$A_2 = \overline{T^{-1}}B_2, \tag{12}$$
$$A_3 = \overline{T^{-1}}\left(B_1D_1 + B_2D_2\right), \tag{13}$$

with

$$B_1 = \beta\sqrt{(1+\gamma)\int_0^1\left[I_v\left(\sqrt{\frac{\lambda}{\epsilon\left(\overline{u}_\star\right)}}x\right)\right]^2 xdx}, \tag{14}$$

$$B_2 = \sqrt{(1+\gamma^{-1})\sum_{m=1}^{\infty}\frac{1}{\sigma_m^2}dx}, \tag{15}$$

for some $\lambda > 0$ and $\gamma > 0$. The function $I_v(\cdot)$ in (14) is the modified Bessel function of first kind. The term $\beta$, in (14), is

$$\beta = \frac{\sqrt{1+b^2}}{\left|bI_v\left(\sqrt{\frac{\lambda}{\epsilon(\overline{u}_\star)}}\right) - \sqrt{\frac{\lambda}{\epsilon(\overline{u}_\star)}}I_{v+1}\left(\sqrt{\frac{\lambda}{\epsilon(\overline{u}_\star)}}\right)\right|}, \tag{16}$$

for some $b > 0$, chosen for (16) to be finite. For each $m \in \mathbb{N}$,

$$\sigma_m = \epsilon\left(\overline{u}_\star\right)\mu_m^2 + \lambda, \tag{17}$$

where $\mu_m$ are the positive roots of

$$\mu J_v'(\mu) + [b - v]J_v(\mu) = 0, \tag{18}$$

in ascending order, and $J_v(\cdot)$ is the Bessel function of first kind with $v = n/2 - 1$. The positive scalars $\overline{T^{-1}}$ and $\overline{T}$ are defined as

$$\overline{T} = 1 + \|K(x,s)\|_2, \tag{19}$$
$$\overline{T^{-1}} = 1 + \|L(x,s)\|_2. \tag{20}$$

Functions $K(x,s)$ and $L(x,s)$ take values on $\mathbb{R}$ and are defined on the unit square $\mathcal{S} := \{(x,s) : 0 \le x, s \le 1\}$ as follows

$$K(x,s) = -s\frac{\lambda}{\epsilon\left(\overline{u}_\star\right)}\frac{I_1\left[\zeta(x,s)\right]}{\zeta(x,s)}, \tag{21}$$

$$L(x,s) = -s\frac{\lambda}{\epsilon\left(\overline{u}_\star\right)}\frac{J_1\left[\zeta(x,s)\right]}{\zeta(x,s)}, \tag{22}$$

with

$$\zeta(x,s) = \sqrt{\frac{\lambda}{\epsilon\left(\overline{u}_\star\right)}\left(s^2 - x^2\right)}. \tag{23}$$

Constants $D_1$ and $D_2$ are

$$D_1 = \frac{1}{\epsilon\left(\overline{u}_\star\right)^2}, \tag{24}$$

$$D_2 = \frac{1}{\epsilon\left(\overline{u}_\star\right)} \overline{T^{-1}} \max_{x\in[0,1]} K(x,1). \tag{25}$$

Finally, a third positive scalar $A_1$ is defined as

$$A_1 = \overline{T^{-1}} \cdot \overline{T}. \tag{26}$$

Now, with $A_1$, $A_2$, and $A_3$ defined, the main result regarding the stability of the estimation error system can be stated.

*Theorem 1:* Consider the estimation error system in (9)-(11) with initial conditions $\widetilde{u}_0(x) \in C^2\left([0,1]\right)$, for $n \le 4$, and observer gains chosen as

$$P(x) = \frac{\lambda}{z(x)}\left[\frac{\lambda}{\epsilon\left(\overline{u}_\star\right)}\frac{I_2\left(z(x)\right)}{z(x)} + (2+b-n)I_1\left(z(x)\right)\right], \tag{27}$$

$$Q = b + \frac{\lambda}{2\epsilon\left(\overline{u}_\star\right)}, \tag{28}$$

for some $\lambda, b > 0$, and $z(x)$ defined as

$$z(x) = \sqrt{\frac{\lambda}{\epsilon\left(\overline{u}_\star\right)}\left(1-x^2\right)}. \tag{29}$$

Then, it follows that

$$\begin{aligned}\|\widetilde{u}(\cdot,t)\|_2 <& A_1\sqrt{\frac{\exp\left[-\sigma_1 t\right]}{2-\exp\left[-\sigma_1 t\right]}}\|\widetilde{u}_0(\cdot)\|_2\\ &+ A_3|\epsilon_1||\delta\overline{u}(t)| \max_{\tau\in[0,t]} |f(\tau)|,\\ &+ A_2|\epsilon_1||\delta\overline{u}(t)| \max_{\tau\in[0,t]} |g(\tau)|\end{aligned} \tag{30}$$

with $\sigma_1$, $A_1$, $A_2$ and $A_3$ computed following Definition 1, and

$$g(t) = \max_{x\in[0,1]} \left|h(x,t)\right|, \tag{31}$$

$$h(x,t) = \frac{1}{x^{n-1}}\left[x^{n-1}u_x(x,t)\right]_x, \tag{32}$$

provided that $f(t) \in C^2(\mathbb{R}_+)$ and $h(x,t) \in C^1(\mathbb{R}_+ \times [0,1])$.

*Proof:* The proof of Theorem 1 is a results of the next two lemmas. First, in Lemma 2, we derive an invertible transformation $T$ that maps the estimation error system (9)-(11) to an auxiliary system; the *target* system. Then, in Lemma 3, we derive an ISS result for the *target* system. The ISS result for the target system and the invertibility of the transformation $T$ imply the ISS property (30) for the estimation error system. ■

*Remark 1:* The regularity condition $h(x,t) \in C^1(\mathbb{R}_+ \times [0,1])$ is a condition on the solutions of the nonlinear PDE (1)-(3). Whenever the system (1)-(3) satisfies this condition, or the additional requirements, is beyond the scope of this paper.

*Lemma 2:* There exists a bounded and invertible transformation $T : \mathcal{L}^2([0,1]^2) \to \mathcal{L}^2([0,1]^2)$ of the form

$$T[v] = v(x,t) - \int_x^1 K(x,s)v(s,t)dx, \tag{33}$$

with inverse

$$T^{-1}[v] = v(x,t) + \int_x^1 L(x,s)v(s,t)dx, \tag{34}$$

and $K(x,s)$, $L(x,s)$ defined in (21) and (22), which maps the error system (9)-(11) to the *target* system

$$\begin{aligned}w_t(x,t) =& \frac{\epsilon\left(\overline{u}_\star\right)}{x^{n-1}}\left[x^{n-1}w_x(x,t)\right]_x - \lambda w(x,t)\\ &+ \epsilon_1\delta\overline{u}(t)T^{-1}\left[\frac{1}{x^{n-1}}\left[x^{n-1}u_x(x,t)\right]_x\right]\\ &- \epsilon_1\delta\overline{u}(t)T^{-1}\left[K(x,1)\right]u_x(1,t),\end{aligned} \tag{35}$$

for $x \in (0,1)$, $t > 0$, with boundary conditions

$$w_x(0,t) = 0, \tag{36}$$

$$w_x(1,t) = -bw(1,t) - \frac{\epsilon_1\delta\overline{u}(t)}{\epsilon\left(\overline{u}_\star\right)}u_x(1,t), \tag{37}$$

and initial conditions $w_0(x) = T^{-1}[\widetilde{u}_0(\cdot)]$.

*Proof:* Lemma 2 is actually an special case of the results in [9], [10] and the proof follows the same steps. ■

*Lemma 3:* Consider $w(x,t)$ satisfying equation (35) with boundary conditions (36)-(37) and initial conditions $w_0(x) \in C^2\left([0,1]\right)$. Then, $w(x,t)$ satisfies the following inequality

$$\begin{aligned}\|w(\cdot,t)\|_2 \le& \sqrt{\frac{\exp\left[-\sigma_1 t\right]}{2-\exp\left[-\sigma_1 t\right]}}\|w_0(\cdot)\|_2\\ &+ |\epsilon_1||\delta\overline{u}(t)|\left(B_1D_1 + B_2D_2\right) \max_{\tau\in[0,t]} |f(\tau)|\\ &+ |\epsilon_1||\delta\overline{u}(t)|B_2 \max_{\tau\in[0,t]} |g(\tau)|,\end{aligned} \tag{38}$$

for $t > 0$, with $g(t)$ defined in (31) and $\sigma_1, B_1, B_2, D_1, D_2$ from Definition 1.

*Proof:* This lemma is a particular, but singular, case of the ISS results in [6], [7]. This singularity originates from the radial diffusion operator and force us to certify that the ISS results are still valid. There are two items that we need verify. First, we need to check that the singular Sturm–Liouville problem

$$\begin{aligned}\epsilon\left(\overline{u}_\star\right)\frac{d}{dx}\left[x^{n-1}\frac{d\phi_m}{dx}(x)\right] - \lambda x^{n-1}\phi_m(x)&\\ = -\sigma_m x^{n-1}\phi_m(x),&\end{aligned} \tag{39}$$

for $x \in (0,1)$ with boundary conditions

$$\phi_m'(0) = 0, \tag{40}$$

$$\phi_m'(1) + b\phi_m(1) = 0, \tag{41}$$

has all the same properties as a regular Sturm–Liouville problem; this is in fact true and the proof is in Lemma 4. Then, we need to verify the convergence of the series

$$\sum_{m=1}^{\infty}\frac{1}{\sigma_m}\max_{x\in[0,1]}|\phi_m(x)|, \tag{42}$$

where $\{\sigma_m, \phi_m(x)\}$ are the eigenvalues and eigenfunctions of (43) - (45). We show in Lemma 5 that the series (42) is in fact convergent for $n \le 4$. ■

*Lemma 4:* The singular Sturm-Liouville problem

$$\epsilon\left(\overline{u}_{\star}\right)\frac{d}{dx}\left[x^{n-1}\frac{d\phi_m}{dx}(x)\right]-\lambda x^{n-1}\phi_m(x) = -\sigma_m x^{n-1}\phi_m(x), \tag{43}$$

for $x \in (0,1)$ with boundary conditions

$$\phi_m'(0)=0, \tag{44}$$

$$\phi'(1)+b\phi(1)=0, \tag{45}$$

preserves the same elementary properties as regular Sturm-Liouville problems, namely:

1) eigenvalues $\sigma_m \in \mathbb{R}$ form an infinite, increasing sequence $\sigma_1<\sigma_2<\cdots<\sigma_m<\dots$, with $\sigma_m\to\infty$ as $m\to\infty$,
2) to each eigenvalue $\sigma_m$ corresponds exactly one real valued eigenfunction $\phi_m(x)\in C^2([0,1])$, satisfying (43) and boundary conditions (44), (50). Furthermore, eigenfunctions $\phi_m(x)$ form an orthonormal basis of $\mathcal{L}^2\left((0,1);x^{m-1}dx\right)$.

*Proof:* The solution to equation (43) is of the form

$$\phi(x)=c_1\phi_1(x)+c_2\phi_2(x). \tag{46}$$

Here, functions $\phi_1(x)$ and $\phi_2(x)$ are two independent solutions of (43) and a closed-form expression is available

$$\phi_1(x)=x^{-v}J_v(\mu x), \tag{47}$$

$$\phi_2(x)=x^{-v}Y_v(\mu x), \tag{48}$$

where $J_v$ and $Y_v$ are the Bessel functions of first and second kind of order $v$ and

$$\mu=\left(\frac{\sigma-\lambda}{\epsilon}\right)^{1/2}, v=\frac{n}{2}-1. \tag{49}$$

Boundary condition (44) is to be replaced with a condition on the regularity of solutions; specifically, we require $\phi(x)$ to be square integrable, i.e.

$$\int_0^1\phi(x)^2dx<\infty. \tag{50}$$

The regularity condition implies that $c_2=0$. Boundary condition (45) provides a characteristic equation, that is, an equation that defines the eigenvalues of the system

$$\mu J_v'(\mu)+[b-v]J_v(\mu)=0,\ \text{if } b>0, \tag{51}$$

$$\mu J_{v+1}(\mu)=0,\ \text{if } b=0. \tag{52}$$

Note that the left hand side of equation (51) is the Dini function [18, p.580], that is, $D_{b,v}(\mu):=\mu J_v'(\mu)+[b-v]J_v(\mu)$. Let $\mu_{b,v,m}$ denote the $m$-th positive zero of $D_{b,v}$, then $\mu_{b,v,m}$ satisfies (51) and defines the value of the $m$-th eigenvalue for the singular Sturm-Liouville problem for $b>0$, that is

$$\sigma_m=\epsilon\left(\overline{u}_{\star}\right)\mu_{b,v,m}^2+\lambda. \tag{53}$$

The corresponding $m$-th eigenfunction is

$$\phi_m(x)=c_{1,m}x^{-v}J_v\left(\mu_{b,v,m}x\right), \tag{54}$$

and $c_{1,m}$ is chosen to normalize (59), that is

$$c_{1,m}=\frac{\sqrt{2}}{J_v\left(\mu_{b,v,m}\right)}. \tag{55}$$

Note that $c_{1,m}$ is well defined in (55), since zeros of the Dini function can not be zeros of Bessel functions because the (non zero) zeros of $J_v, J_{v+1}$ never coincide. The $m-$th positive zero $\mu_{v+1,m}$ of $J_{v+1}$, and *zero* itself, are solutions of (52). Thus for $b=0$ and $n\le 2$ the eigenvalues of the problem are

$$\sigma_m=\lambda,\ \text{for } m=1 \tag{56}$$

$$\sigma_m=\epsilon\mu_{v+1,m-1}^2+\lambda,\ \text{for } m>1, \tag{57}$$

and, for $n>2$ the eigenvalues of the problem are

$$\sigma_m=\epsilon\mu_{v+1,m}^2+\lambda, \tag{58}$$

The corresponding eigenfunctions are

$$\phi_m(x)=c_{1,m}x^{-v}J_v\left(\mu_{v+1,m}x\right), \tag{59}$$

and $c_{1,m}$ is chosen to normalize (59), that is

$$c_{1,m}=\frac{\sqrt{2}}{J_v\left(\mu_{v+1,m}\right)}. \tag{60}$$

The statements in the lemma follow directly from properties of the Bessel functions:

1) For $b>0$ and $v>-1$, the positive zeros of $D_{b,v}$ are real and form an infinite increasing sequence [18, p.580, p.597].[1] Similarly, for $v>-1$, the positive zeros of $J_v$ are real and form an infinite and increasing sequence [18, p. 479]. Thus, from (49) and (51), (52) it follows that the eigenvalues of the singular Sturm-Liouville problem are real, positive and form an infinite and increasing sequence.
2) Let $\{\mu_{b,v,m}\}$ and $\{\mu_{v+1,m}\}$ be the zeros of $D_{b,v}$ and $J_{v+1}$, respectively. The fact that $\{c_1x^{-v}J_v[\mu_{b,v,m}x]\}$ and $\{c_1x^{-v}J_v[\mu_{v+1,m}x]\}$ are orthonormal basis of $\mathcal{L}^2\left((0,1);x^{n-1}dx\right)$ is known [18, Chapter 18][2], [19, Theorem 3].

■

*Lemma 5:* Let $\{\sigma_m\}$ and $\{\phi_m(x)\}$ be the eigenvalues and eigenfunctions of the singular Sturm-Liouville problem in Lemma 4. Then, it holds that

$$\sigma_1>0, \tag{61}$$

and, in the case $n\le 4$

$$\sum_{m=1}^{\infty}\frac{1}{\sigma_m}\max_{x\in[0,1]}|\phi_m(x)|<\infty. \tag{62}$$

*Proof:* For any $v>-1/2$, the $m$-th positive zero $\mu_{\alpha,m}$ of the Bessel function $J_\alpha$ is lower bounded [20] as follows

$$\mu_{\alpha,m}>m\pi-\frac{\pi-1}{2}+\alpha. \tag{63}$$

[1] Although these properties hold for $v\ge-1$, the exposition in [18] assumes $v\ge-1/2$, yet for our problem $n\ge1$ implies $v>-1/2$.

[2] For the space $\mathcal{L}^2\left((0,1);xdx\right)$ the orthogonal basis $\{c_1J_v[\mu_m x]\}$ is known as the Fourier-Dini series

Let $\mu_{b,\alpha,m}$ be the $m$-th positive zero of the Dini function $D_{b,\alpha}$. From the Dixon's theorem on interlacing zeros of Dini functions [18, p.480] it follows that

$$\mu_{b,\alpha,m} > \mu_{b',\alpha,m-1}, \tag{64}$$

If we let $b' = 0$, zeros $\mu_{b',\alpha,m-1}$ are actually the positive zeros of $J_{\alpha+1}$, and *zero* itself, therefore

$$\mu_{b,\alpha,1} > 0, \tag{65}$$

$$\mu_{b,\alpha,m} > \mu_{\alpha+1,m-1}, \text{ for } m > 1 \tag{66}$$

In the case $b = 0$, equation (63) implies

$$\sigma_m > \lambda, \text{ for } m \in \{1, 2\}, \tag{67}$$

$$\sigma_m > \epsilon(m-2)^2\pi^2 + \lambda, \text{ for } m > 2. \tag{68}$$

In the case $b > 0$,

$$\sigma_m > \lambda, \text{ for } m \in \{1, 2, 3\}, \tag{69}$$

$$\sigma_m > \epsilon(m-3)^2\pi^2 + \lambda, \text{ for } m > 3. \tag{70}$$

Clearly the first inequality in the lemma holds.
We use a known bounds on Bessel functions

$$\max_{x\in[0,1]} |\phi_m(x)| \le \frac{\sqrt{2}}{J_v(\mu_{b,v,m})}\mu_{b,v,m}^v. \tag{71}$$

For any $m > 0$ we have $J_v(\mu_{b,v,m}) \ne 0$, and thus all terms in the series are bounded. This allows us to neglect the first $M$ terms of the series, for any $M > 0$, and concern only about the convergence of the tail. From this observation and from (68) and (70) it follows that convergence of

$$\sum_{m=M}^{\infty} \frac{1}{J_v(\mu_{b,v,m+3})} \frac{\mu_{b,v,m+3}^v}{m^2}, \tag{72}$$

implies convergence of the original series (62). Let

$$a_m = \frac{1}{\mu_{b,v,m+3}^{1/2} J_v(\mu_{b,v,m+3})}, \tag{73}$$

$$b_m = \frac{\mu_{b,v,m+3}^{v+1/2}}{m^2}, \tag{74}$$

It can be verified that

$$\inf_{m>0} \left| \mu_{b,v,m+3}^{1/2} J_v(\mu_{b,v,m+3}) \right| > 0, \tag{75}$$

thus, the sequence $\{|a_m|\}$ is bounded. From the asymptotic location of zeros of Bessel functions, there is $M > 0$ such that for $m > M$ we have

$$|b_m| < \frac{((m+3) + (v+1)/2)^{v+1/2}}{m^2}\pi^{v+1/2}. \tag{76}$$

Using a one-sided comparison test, the convergence of the series is guarantee to converge if $v + 1/2 < 2$; this is the case if $n \le 4$. Boundedness of $\{|a_m|\}$ and convergence of $\{|b_m|\}$ implies that the original series converges for $n \le 4$. ■

Now that the proof of the main result is complete, we can proceed with a brief description on how this results is applied to state estimation for lithium-ion batteries from the SPM.

## IV. The Single Particle Model

A simple electrochemical model accounting for some of the main dynamic phenomena in lithium-ion batteries is the SPM [21]. The model includes a pair of diffusion equations describing the diffusion of lithium ions in the intercalation sites of active materials in the electrodes

$$\frac{\partial c_{\mathrm{s},\pm}}{\partial t}(r,t) = \frac{D_{\mathrm{s}}(\overline{c_{\mathrm{s}}}_{\pm}(t))}{r^{n-1}} \frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial c_{\mathrm{s},\pm}}{\partial r}(r,t)\right], \tag{77}$$

for $r \in (0, R_{\mathrm{p},\pm})$, $t > 0$, $n \in \{1, 2, 3\}$, with boundary conditions

$$\frac{\partial c_{\mathrm{s},\pm}}{\partial r}(0,t) = 0, \tag{78}$$

$$D_{\mathrm{s},\pm}(\overline{c_{\mathrm{s}}}(t))\frac{\partial c_{\mathrm{s},\pm}}{\partial r}(R_{\mathrm{p},\pm}, t) = -j_{\pm}(t). \tag{79}$$

The diffusion coefficients $D_{s,\pm}$ are functions of the average concentration of lithium ions $\overline{c_{\mathrm{s}}}_{,\pm}(t)$. The terms $j_{\pm}(t)$ are molar fluxes of lithium ions, i.e., the rate of lithium entering or exiting the intercalation sites. The parameters $R_{p,\pm}$ are the average, or representative, radii of the particles. We view (77) - (79) as a dynamic system with states $c_{\mathrm{s},\pm}(r,t)$, input $j_{n,\pm}(t)$ and output $c_{\mathrm{ss},\pm}(t) = c_{\mathrm{s},\pm}(R_{\mathrm{p},\pm}, t)$. Molar fluxes are computed as a proportion of the current $I(t)$ (per unit area) applied to the lithium-ion cell

$$j_-(t) = \frac{nR_{p,-}}{\epsilon_{\mathrm{s},-}FL_-}I(t), \; j_+(t) = -\frac{nR_{p,+}}{\epsilon_{\mathrm{s},+}FL_+}I(t). \tag{80}$$

Where $\epsilon_{s,\pm}$ are the volume fractions of active material in the electrode, $L_{\pm}$ are the lengths of the electrodes and $F$ is the Faraday constant. Overpotentials $\eta_{\pm}(t)$ are computed by solving a set of nonlinear algebraic equations (in terms of $j_{\pm}$ and $c_{\mathrm{ss},\pm}(t)$)

$$j_{\pm}(t) = \frac{i_{0,\pm}(t)}{F}\left[e^{\frac{\alpha_{\mathrm{a}}F}{RT}\eta_{\pm}(t)} - e^{-\frac{\alpha_{\mathrm{c}}F}{RT}\eta_{\pm}(t)}\right], \tag{81}$$

$$i_{0,\pm}(t) = k_{\pm}\left[c_{\mathrm{ss},\pm}(t)\right]^{\alpha_{\mathrm{c}}}\left[c_{\mathrm{e}}\left(c_{\mathrm{s,max},\pm} - c_{\mathrm{ss},\pm}(t)\right)\right]^{\alpha_{\mathrm{a}}}, \tag{82}$$

where $k_{\pm}$ are (effective) reaction rates, $c_e$ is the concentration of lithium-ions in the electrolyte (assumed to be constant), $T$ is the mean temperature in the cell, and $R$ is the gas constant. Electric potentials in the electrodes are computed from

$$\phi_{\mathrm{s},\pm}(t) = \eta_{\pm},(t) + U_{i^-}(c_{\mathrm{ss},\pm}(t)) + R_{\mathrm{f},\pm}Fj_{\pm}(t), \tag{83}$$

where $U_{\pm}$ are open-circuit potentials. Finally, the measured voltage in the cell is the difference between the positive and negative electric potential,

$$V(t) = \phi_{\mathrm{s},+}(t) - \phi_{\mathrm{s},-}(t). \tag{84}$$

Concentrations $c_s(r,t)$ are positive and bounded by $c_{\mathrm{s,max},\pm}$, where the possible values of $c_{\mathrm{s,max},\pm}$ depend on the specific active material. The current applied to the cell is bounded to keep the concentration within these bounds. For an experimental determination of the dependence of diffusion on the mean concentration of lithium-ions in the electrodes, for a particular material, one can see the results in [5]. An observer, based on the SPM, can be derived to estimate the concentration of lithium ions in the electrode; and thus, estimating the state of charge.

## V. Observer for the Single Particle Model

We assume $c_{\mathrm{ss},\pm}(t)$ can be recover perfectly from measurements of current and voltage. Thus, the problem of estimating $c_{\mathrm{s},\pm}(r,t)$ from known values of $j_n(t)$ and $c_{\mathrm{ss},\pm}(t)$ can be solved using the analysis in the previous sections. We consider an observer in the form

$$\frac{\partial \widehat{c_{\mathrm{s},\pm}}}{\partial t}(r,t) = \frac{D_{\mathrm{s}}\left(\overline{c_{\mathrm{s}}}_{\pm,\star}\right)}{r^{n-1}} \frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial \widehat{c_{\mathrm{s},\pm}}}{\partial r}(r,t)\right] \tag{85}$$

$$+ P(r)\left[c_{\mathrm{ss},\pm}(t) - \widehat{c_{\mathrm{ss},\pm}}(t)\right], \tag{86}$$

for $r \in (0, R_{\mathrm{p},\pm})$, $t > 0$, $n \in \{1,2,3\}$, with boundary conditions

$$\frac{\partial \widehat{c_{\mathrm{s},\pm}}}{\partial r}(0,t) = 0, \tag{87}$$

$$D_{\mathrm{s},\pm}\left(\overline{c_{\mathrm{s}}}_{\pm,\star}\right)\frac{\partial \widehat{c_{\mathrm{s},\pm}}}{\partial r}(R_{\mathrm{p},\pm},t) = -j_{,\pm}(t) \tag{88}$$

$$+ Q\left[c_{\mathrm{ss},\pm}(t) - \widehat{c_{\mathrm{ss},\pm}}(t)\right]. \tag{89}$$

Gains $P(r)$ and $Q$ are the ones appearing in Theorem 1; with the parameters of the model and after proper scaling of the domain. With this observer, the estimation error is bounded as follows

$$\begin{aligned}\|\widetilde{c_{\mathrm{s}}}_{\pm}(\cdot,t)\| <& A_{1,\pm}\sqrt{\frac{\exp\left[-\sigma_1 t\right]}{2-\exp\left[-\sigma_1 t\right]}}\|\widetilde{c_{\mathrm{s}}}_{\pm}(\cdot,0)\| \\ &+ |D_{s,\pm,1}||\delta\overline{c_{\mathrm{s}}}_{\pm}(t)|A_{2,\pm}\max_{\tau\in[0,t]}|g(\tau)| \\ &+ |D_{s,\pm,1}||\delta\overline{c_{\mathrm{s}}}_{\pm}(t)|A_{3,\pm}\max_{\tau\in[0,t]}\end{aligned} \tag{90}$$

with

$$g_{\pm}(t) = \max_{r\in[0,1]}\left|\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial c_{\mathrm{s},\pm}}{\partial r}(r,t)\right]\right| \tag{91}$$

and

$$\delta\overline{c_{\mathrm{s}}}_{\pm}(t) = \overline{c_{\mathrm{s}}}_{\pm}(t) - \overline{c_{\mathrm{s}}}_{\pm,\star}. \tag{92}$$

Constants $A_{1,\pm}$, $A_{2,\pm}$ and $A_{3,\pm}$ are the ones in Definition 1, using the with the parameters of this model.

*Remark 2:* Note that if $D_{\mathrm{s},\pm}$ is constant, then $D_{\mathrm{s},\pm,1} = 0$, thus recovering the convergence properties of the boundary observer for diffusion equations with constant parameters.

## VI. Conclusions

This paper discusses the problem of state estimation for a diffusion equation with a diffusion coefficient depending on the value of the spatial average of the state. The main contribution of this paper is the derivation of bounds in the estimation error that arises in this particular situation. The derivation of these bounds follow recent results on the input-to-state stability of one-dimensional parabolic equations. The main technical challenge is to verify that conditions and assumptions for the ISS results to hold are valid for radial diffusion equations in $n$-dimensional balls.


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