System Analysis and Optimization of Human-Actuated Dynamical Systems

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Abstract— **This paper investigates dynamical systems where system inputs are induced by human behavior. In particular, we consider linear time-invariant systems with a stochastic discrete choice actuation model. We are motivated by increasingly important cyber-physical-social systems (CPSS), such as smart mobility, smart energy, and smart cities. Existing literature regarding random dynamical systems (RDS) predominantly considers additive noise models with well-defined probability distributions. Furthermore, the role of human interactions is usually considered a disturbance. The closed-loop system must not be designed explicitly for this disturbance, but must be robust to it instead. This paper adds two original contributions to the existing literature. First, we integrate Discrete Choice Models (DCM) from behavioral economics into dynamical systems to incorporate human decision making, yielding a Dynamical System with Discrete Choice Models (DSDCM). System inputs are triggered by human actuators, who act selfishly by taking actions that maximize their own utility functions. Second, we formulate a convex optimization problem for DSDCM that seeks to incentivize human decision making to achieve a system-wide objective. Finally, we apply DSDCM in the context of demand response and provide potential directions for future work.**

I. INTRODUCTION

This paper investigates dynamical systems where system inputs are induced by human behavior. We mathematically model these systems using discrete choice models (DCM) from behavioral economics for the actuator model, yielding a Dynamical System with Discrete Choice Model (DSDCM). We analyze DSDCM for stationarity, stability, and controllability. Moreover, we derive a convex optimization problem for control synthesis, which incentivizes human decision making to achieve a given objective. Our proposed models will prove useful to system operators in coping with challenges that arise due to randomness resulting from human components, such as in cyber-physical-social systems (CPSS). In addition, optimal system operations that result from solving the convex optimization problem introduced in this paper may provide economic benefits to system operators in various contexts.

Existing literature on dynamical systems with random inputs provides multiple approaches to addressing uncertainties. These approaches are useful in systems where human inputs are regarded as sources of noise. The intersection between probability theory and dynamical systems is referred to as random dynamical systems (RDS) in this paper. RDS represent systems that are perturbed by noise, and have been thoroughly scrutinized throughout the literature [1]. Various concepts in stochastic processes, e.g. Markov chains and martingales, have been used to address randomness in dynamical systems [2]–[4]. Optimal control schemes for RDS, concerning additive noise models with well-defined probability distributions, have also been extensively studied [5]–[8]. We can also directly apply discrete-time RDS with statistical signal processing techniques when RDS are viewed as random processes [9].

The aforementioned literature provides fundamental theories that are necessary to address dynamical systems involving human components. In the field of robotics, human-in-the-loop dynamical systems with human actuators have been analyzed [10]. However, human roles have always been restricted to improving system performances [11] in this area. This paper extends the existing literature on dynamical systems involving human actuators by integrating DCM into dynamical systems to handle a set of possible behaviors, or alternatives [12], [13]. Individual human behavior is random by nature, and is determined with only his or her interest in mind. DCM evaluate the probabilities of possible alternatives and provide mathematical formalism that allows us to model human behavior in dynamical systems. With the model for human behavior fixed, we can develop control and optimization schemes for the dynamical systems where the system inputs are induced by human behavior. Determining attributes of alternatives that impact probability distributions of possible behaviors is of specific interest in this paper. We offer a different perspective on studying dynamical systems with DCM. Dynamical systems with DCM usually take attributes as given constants of the system. Here, we treat attributes as decision variables of an optimization problem.

Attribute optimization in DCM can be interpreted as pricing optimization in settings where prices can be represented by attributes that are dynamically adjustable. Pricing optimization has been extensively studied in various contexts [14]–[16]. However, existing approaches are not tailored to the specific needs of our problem, so we propose to solve the problem of optimizing over the DCM attributes using convex optimization techniques. It is possible to draw parallels between DSDCM and stochastic hybrid system frameworks. DSDCM can be represented as a stochastic hybrid system where discrete states represent human decisions and transitions between states occur in continuous time, induced by incentive controls. This idea of extending CPSS modeling in

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the context of stochastic hybrid systems remains as future work.

We add two original contributions. First, we present a mathematical framework and system analysis of DSDCM. Second, we formulate a convex optimization problem for the control of DSDCM, and derive a closed-form, analytic expression for the gradient of the objective function, yielding an algorithm that solves the control problem. Additionally, we demonstrate the applicability of the DSDCM framework on a demand response example.

This paper is organized in the following manner: Section II presents the DSDCM system analysis. Section III presents the convex optimization problem, and proposes an algorithm for solving the optimization problem effectively. Section IV presents a practical application of DSDCM in the context of demand response. Conclusions are drawn in Section V.

II. System Analysis of Dynamical Systems with Discrete Choice Models

In this section, we introduce a system overview of DSDCM and mathematically represent it as a state space model. We further analyze the system for equilibria, stability, and controllability.

A. System Overview

Consider a dynamical system $x(k+1) = Ax(k) + Bu(k)$, where the input $u(k)$ is determined by human choices. Humans select from a discrete set of choices, called alternatives. The probability of selecting a given alternative depends on a utility function. These utility functions are functions of controllable variables, e.g. a price incentive, and uncontrollable variables, e.g. the weather. Such models are known as discrete choice models in the behavioral economics literature. This framework characterizes many CPSS. For example, the dynamics of a store's inventory is governed by consumer purchases. A consumer purchase is a random event, which can depend on store-controlled variables, e.g. a sale, and non-store-controlled variables, e.g. the weather. Other examples include on-demand mobility services, and demand response programs in power systems, which we explore in Section IV. We seek a rigorous mathematical framework to analyze and manage these CPSS.

Mathematically, we formulate DSDCM in the canonical linear time-invariant discrete-time state space representation as

$$x(k+1) = Ax(k) + Bu(k), \quad (1)$$

where $x(k)$ denotes the system state and $u(k)$ is the random system input obtained by DCM. The system inputs are constrained, i.e. $u(k) \in \{u_1, u_2, \cdots, u_J\}$, to a finite set of J alternatives. Each system input alternative has a specific utility function, and an alternative is chosen when its utility is higher than that of others. For the j-th alternative, $j \in \{1, 2, \cdots, J\}$, the utility function is

$$U_j = f_j\left(z(k)\right) \doteq \beta_j^\top z(k) + \gamma_j^\top w(k) + \beta_{0j} + \epsilon_j, \quad (2)$$

where $z(k)$ is a set of controllable inputs, $w(k)$ is a set of uncontrollable inputs at time-step k, β_j and γ_j are sets of parameters for the controllable inputs and uncontrollable inputs, respectively, β_{j0} is an alternative specific constant, and ϵ_j accounts for all unspecified errors. Fig. 1 illustrates this system.

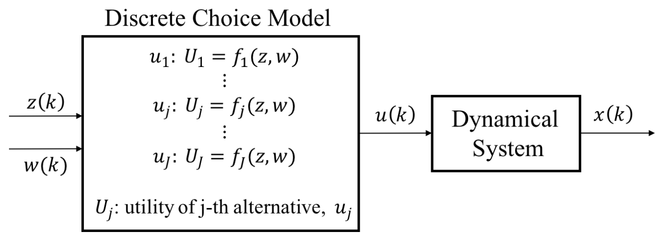

Fig. 1. Block diagram of Dynamical Systems with Discrete Choice Model where $z(k)$ is a set of controllable inputs, $w(k)$ is a set of uncontrollable inputs, and $u(k) \in \{u_1, u_2, \cdots, u_J\}$.

The distribution of the error terms ϵ_j determines the probability mass function of $u(k)$. The two most popular models of the distributions of ϵ_j for every j are *i.i.d* extreme value distribution and Gaussian distribution, which correspond to (multinomial) logit models and probit models, respectively. In DSDCM, errors are assumed to follow the *i.i.d* extreme value distribution because under the logit model, the probability of choosing the j-th alternative can be written

$$\Pr(u(k) = u_j) = \Pr\left[\bigcap_{j \neq i} (U_j > U_i)\right] = \frac{e^{V_j}}{\sum_{i=1}^{J} e^{V_i}}, \quad (3)$$

where $V_j \doteq \beta_j^\top z(k) + \gamma_j^\top w(k) + \beta_{0j}$. We exploit the potential predictive power of the logit model [17], [18] to evaluate human components in CPSS.

B. Stationarity and Boundedness

Assume that the probability of choosing the j-th alternative does not vary over time for all $j \in \{1, 2, \cdots, J\}$ and that independence holds across all time-steps. We can interpret this system as a stochastic process where the system input is an *i.i.d.* random variable. Since the system is a random process, the traditional definition of an equilibrium for deterministic systems does not apply. Instead, the system equilibrium can be analyzed in terms of stationarity, summarized by the following proposition.

Proposition 1: Consider the DSDCM (1)-(3). Assume the inputs $u(k)$ are *i.i.d.* In the limit $k \to \infty$, the states $x(k)$ are wide-sense stationary if $|\lambda_{\max}(A)| < 1$, where $\lambda_{\max}(A)$ denotes the largest eigenvalue of A.

Proof: If $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$, then $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$. For elegance in presentation, let $A = a$ and $B = b$, $a, b \in \mathbb{R}$, without loss of generality. The system is said to be wide-sense stationary if the first moment of the states does not vary with respect to time and the autocovariance between states at two time-steps can be written as a function of only the time difference.

The closed-form, analytic solution to the discrete time-invariant dynamical system with random variable input is

$$x(k) = a^k x(0) + \sum_{i=0}^{k-1} a^{k-i-1} b u(i). \tag{4}$$

In the limit $k \to \infty$, the expected value of the state (4) is

$$\begin{aligned}\lim_{k\to\infty} \mathbb{E}[x(k)] &= \lim_{k\to\infty} \mathbb{E}\left[\sum_{i=0}^{k-1} a^{k-i-1} b u(i)\right] \\ &= b\mu \lim_{k\to\infty} \sum_{i=0}^{k-1} a^{k-i-1} \\ &= \frac{b\mu}{1-a},\end{aligned} \tag{5}$$

where μ is the mean of $u(k)$ over all time-steps k. The summation in the proof converges if $|a| < 1$ in the scalar case and if $|\lambda_{\max}(A)| < 1$ in the general n-dimensional case. This indicates that the mean is constant over time.

The covariance of the analytical solution with respect to the two time-steps k_1 and k_2, $k_1 \le k_2$, is

$$\begin{aligned} &C_{xx}(k_1, k_2) \\ &= \mathrm{Cov}\left(b \sum_{i=0}^{k_1-1} a^{k_1-i-1} u(i),\ b \sum_{j=0}^{k_2-1} a^{k_2-j-1} u(j)\right) \\ &= b^2 \sum_{i=0}^{k_1-1} a^{k_1+k_2-2i-2} \sigma^2 \\ &= \frac{b^2 a^\tau \sigma^2}{1-a^2},\end{aligned} \tag{6}$$

where σ is the variance of $u(k)$ over all time-steps k, and τ is the time delay, i.e. $\tau \doteq k_2 - k_1$.

The aforementioned results can be extended to the general n-dimensional case by replacing the denominator with the matrix inverse, exponentiation of a and b with the corresponding matrix exponentials, and using the identity matrix in place of 1. ■

If the probability distribution of $u(k)$ varies over time, then wide-sense stationarity of the system is not guaranteed. However, the mean and variance of $x(k)$ can be bounded from above.

Proposition 2: In the limit $k \to \infty$, the absolute value of the expectation and the variance of the above system are upper-bounded by $\left|\frac{b\mu_{\max}}{1-a}\right|$ and $\frac{b^2\sigma^2_{\max}}{1-a^2}$, respectively, where $\mu_{\max}$ is the maximum mean, and $\sigma^2_{\max}$ is the maximum variance of the input random variables over all time-steps k.

Proof: In the limit $k \to \infty$, the absolute value of the expected value of the analytic solution to the state (4) is

$$\begin{aligned}\lim_{k\to\infty} |\mathbb{E}[x(k)]| &= \lim_{k\to\infty} \left|\mathbb{E}\left[\sum_{i=0}^{k-1} a^{k-i-1} b u(i)\right]\right| \\ &\le |b\mu_{\max}| \lim_{k\to\infty} \left|\sum_{i=0}^{k-1} a^{k-i-1}\right| \\ &= \left|\frac{b\mu_{\max}}{1-a}\right|,\end{aligned} \tag{7}$$

where $\mu_{\max}$ is the maximum mean of $u(k)$ over all time-steps k. In the limit $k \to \infty$, the variance of (4) is

$$\begin{aligned}\lim_{k\to\infty} \mathrm{Var}\,(x(k)) &= \lim_{k\to\infty} \mathrm{Var}\left(\sum_{n=0}^{k-1} a^{k-i-1} b u(i)\right) \\ &\le b^2 \sigma^2_{\max} \lim_{k\to\infty} a^{2(k-1)} \sum_{i=0}^{k-1} a^{-2i} \\ &= \frac{b^2\sigma^2_{\max}}{1-a^2},\end{aligned} \tag{8}$$

where the second equality follows from the independence of the random variable inputs, and $\sigma^2_{\max}$ is the maximum variance of $u(k)$ over all time-steps k. ■

C. State Mean Analysis

We further investigate the mean state, denoted by $\bar{x}(k)$. It is useful to analyze the mean state, since we often seek to control the mean state. We define the system mean dynamics with respect to $\bar{x}(k)$ as

$$\bar{x}(k+1) = A\bar{x}(k) + B\mu(z(k)), \tag{9}$$

where $z(k)$ is the controlled input in (2) and $\mu(z(k)) \doteq \mathbb{E}[u(k)]$. Fixing the controlled input $z(k) = z_0$ and neglecting uncontrolled inputs, i.e. $w(k) = 0$, the system can be represented by

$$\bar{x}(k+1) = A\bar{x}(k) + B\frac{\sum_{j=1}^J u_j e^{V_j}}{\sum_{i=1}^J e^{V_i}}, \tag{10}$$

where $V_j = \beta_j^\top z_0 + \beta_{0j}$. By replacing $\bar{x}$ with the equilibrium $\bar{x}^{\mathrm{eq}}$, we can calculate the system mean equilibrium in closed-form and write

$$\bar{x}^{\mathrm{eq}} = (I-A)^{-1} B \frac{\sum_{j=1}^J u_j e^{V_j}}{\sum_{i=1}^J e^{V_i}}, \tag{11}$$

assuming that $I - A$ is invertible, i.e. $|\lambda_{\max}(A)| < 1$. Consequently, this equilibrium is also stable.

Under time-varying control inputs $z(k)$, the utility function is

$$V_j = g\left(z(k)\right) \doteq \beta_j^\top z(k) + \beta_{0j}, \tag{12}$$

where we again neglect uncontrolled inputs. This is a non-linear system with respect to the inputs $z(k)$. We therefore linearize $\mu(z(k))$ with respect to $z(k)$ using Taylor series expansion around $z_a \in \mathbb{R}$. Without loss of generality, we simplify the problem to the case where $z(k) \in \mathbb{R}$. Then, we have

$$\begin{aligned}\mu(z(k)) &\approx \mu(z_a) + \left(\left.\frac{\partial \mu(z(k))}{\partial z(k)}\right|_{z(k)=z_a}\right)(z(k) - z_a) \\ &= \mu(z_a) + P(z(k) - z_a),\end{aligned} \tag{13}$$

where

$$P \doteq \left[\sum_{j=1}^J \frac{u_j e^{V_j|_{z_a}}}{\sum_{i=1}^J e^{V_i|_{z_a}}} \left(\sum_{m=1}^J (\beta_j - \beta_m) e^{V_m|_{z_a}}\right)\right]. \tag{14}$$

The new input matrix $\tilde{B}$ is then defined as

$$\tilde{B} \doteq B \begin{bmatrix} P & (\mu(z_a) - Pz_a) \end{bmatrix}, \tag{15}$$

which allows us to write the system representation as

$$\bar{x}(k+1) = A\bar{x}(k) + \tilde{B} \begin{bmatrix} z(k) \\ 1 \end{bmatrix}. \tag{16}$$

Local controllability of the system mean is determined by investigating the column rank of the controllability matrix

$$\mathcal{C} = \begin{bmatrix} \tilde{B} & A\tilde{B} & A^2\tilde{B} & \cdots & A^{n-1}\tilde{B} \end{bmatrix}. \tag{17}$$

Finally, consider the dynamics of the mean of a closed-loop system. This is the dynamics of the above system with the control input

$$z(k) = K\bar{x}(k), \tag{18}$$

where K is a control gain. The equilibrium of the aforementioned closed-loop system is calculated by solving the equation

$$(I-A)\bar{x}^{\text{eq}} \sum_{i=1}^{J} e^{\beta_i^\top K\bar{x}^{\text{eq}}+\beta_{0i}} = B\sum_{j=1}^{J} u_j e^{\beta_j^\top K\bar{x}^{\text{eq}}+\beta_{0j}}, \tag{19}$$

which can also be written as

$$\sum_{j=1}^{J} \left((I-A)\bar{x}^{\text{eq}} - Bu_k\right) e^{\beta_j^\top K\bar{x}^{\text{eq}}+\beta_{0j}} = 0. \tag{20}$$

Note that this equation is nonlinear in $\bar{x}^{\text{eq}}$, and can be numerically solved via, for example, a Newton-Raphson scheme. However, there are no *a priori* guarantees for the existence or uniqueness of the solution.

III. Convex Optimization Framework

In this section, we present a convex optimization framework for optimizing the controllable inputs, the attributes $z(k)$, of DCM.

A. Objective Function

We first consider the state regulation problem, where the objective function is simply the sum of all expected states over a specified time period T. Such objectives can find multiple applications where the states represent errors we seek to minimize. This objective function, although simple in formulation, is not convex in the decision variables

$$Z = \begin{bmatrix} \longleftarrow z_1 \longrightarrow \\ \vdots \\ \longleftarrow z_T \longrightarrow \end{bmatrix} \in \mathbb{R}^{T\times L}.$$

We discuss how to circumvent this complication in the next section. For brevity of expression, we consider the case where there are one sole human actuator, i.e. $N = 1$, a one-dimensional state, i.e. $A = a, a \in \mathbb{R}$, and a time-varying scalar B, i.e. $B(m) = b(m), b(m) \in \mathbb{R}$ for every m. Then, the objective function becomes

$$f(Z) = \sum_{k=1}^{T} \left(\sum_{m=0}^{k-1} a^{k-m-1} b(m)\bar{u}(m) \right), \tag{21}$$

where $\bar{u}(m)$ is the expected value of the random variable $u(m)$. Under the discrete choice model framework and substituting in the extreme value error with logistic regression for $u(m)$, we obtain

$$f(Z) = \sum_{k=1}^{T} \left(\sum_{m=0}^{k-1} a^{k-m-1} b(m) \frac{\sum_{i=1}^{J} e^{\beta_{mi}^\top z_m} u_i(m)}{\sum_{j=1}^{J} e^{\beta_{mj}^\top z_m}} \right), \tag{22}$$

where the attribute coefficient β_{mj} is defined

$$\beta_{mj} \doteq [\beta_{mj_0}, \beta_{mj_1}, \cdots, \beta_{mj_L}]^\top \in \mathbb{R}^{L+1}, \tag{23}$$

and L is the number of controllable attributes. Note that β_{mj_0} contains utilities for alternative specific constants and exogenous variables $w(m)$ that are not controllable.

We can extend the objective function further by adding the sum of variances of the states. Let f_0 denote the objective function with the added variance term. Then, we obtain

$$f_0(Z) = \sum_{k=1}^{T} \left[\sum_{m=0}^{k-1} a^{k-m-1} b(m) \left(\frac{\sum_{i=1}^{J} e^{\beta_{mi}^\top z_m} u_i(m)}{\sum_{j=1}^{J} e^{\beta_{mj}^\top z_m}} \right) + \lambda_k \left(\sum_{\ell=0}^{k-1} a^{2(k-\ell-1)} b^2(\ell)\sigma_\ell^2 \right) \right], \tag{24}$$

where

$$\sigma_\ell^2 \doteq \sum_{i=1}^{J} \frac{e^{\beta_{\ell i}^\top z_\ell}}{\sum_{j=1}^{J} e^{\beta_{\ell j}^\top z_\ell}} \left(u_i(\ell) - \frac{\sum_{p=1}^{J} e^{\beta_{\ell p}^\top z_\ell} u_p(\ell)}{\sum_{q=1}^{J} e^{\beta_{\ell q}^\top z_\ell}} \right)^2, \tag{25}$$

and $\lambda_k > 0$ is the regularization factor associated with the variance term.

B. Convexity Constraints

Each term inside the inner sum of the objective function (22) can be interpreted as a posynomial, which is in general not convex. Therefore, there is no guarantee that global optima exist. We can however enforce convexity by restricting the domain of the search for optima.

Theorem 1: Consider the simpler case where the number of alternatives $J = 2$, i.e. $u_i(k) \in \{0, 1\}$ and we have a binomial logit model. Also, assume the decision variables z_m are scalars for every time-step $m \in \{0, 1, \ldots, T-1\}$. Then, the optimization problem that minimizes the objective function $f(Z)$ in (22) with respect to the attributes $z(k)$ can be formulated as a convex optimization problem if $z_m(\beta_{m0} - \beta_{m1}) \geq \gamma_{m1} - \gamma_{m0}$, $u_0(m) = 0$, and $u_1(m) = 1$. Parameters β_{m0} and β_{m1} are the attribute coefficients and γ_{m0} and γ_{m1} are the exogenous variables for the decision variables z_m for U_{m0} and U_{m1}, respectively.

Proof: In the binomial logit model, the expected value of the input random variable $\bar{u}(m)$ is

$$\bar{u}(m) = \frac{e^{\beta_{m1}z_m+\gamma_{m1}}}{e^{\beta_{m0}z_m+\gamma_{m0}} + e^{\beta_{m1}z_m+\gamma_{m1}}}. \tag{26}$$

Define $\tilde{\beta}_m \doteq \beta_{m0} - \beta_{m1}$ and $\tilde{\gamma}_m \doteq \gamma_{m0} - \gamma_{m1}$. Then, the first derivative of $\bar{u}(m)$ can be written

$$\frac{\partial \bar{u}(m)}{\partial z_m} = \frac{-\tilde{\beta}_m e^{z_m\tilde{\beta}_m+\tilde{\gamma}_m}}{\left(1+e^{z_m\tilde{\beta}_m+\tilde{\gamma}_m}\right)^2}, \tag{27}$$

and the second derivative of $\bar{u}(m)$ can be written

$$\frac{\partial^2 \bar{u}(m)}{\partial z_m^2} = \frac{\tilde{\beta}_m^2 e^{(z_m \tilde{\beta}_m + \tilde{\gamma}_m)}}{(1+e^{(z_m \tilde{\beta}_m + \tilde{\gamma}_m)})^2} \left(\frac{2e^{(z_m \tilde{\beta}_m + \tilde{\gamma}_m)}}{1+e^{(z_m \tilde{\beta}_m + \tilde{\gamma}_m)}} - 1 \right). \tag{28}$$

If $\frac{\partial^2 \bar{u}(m)}{\partial z_m^2} \geq 0$, $\bar{u}(m)$ is convex in z_m [19]. Observe that

$$\frac{\partial^2 \bar{u}(m)}{\partial z_m^2} \geq 0 \iff \frac{2e^{(z_m \tilde{\beta}_m + \tilde{\gamma}_m)}}{1+e^{(z_m \tilde{\beta}_m + \tilde{\gamma}_m)}} - 1 \geq 0 \tag{29}$$

if and only if

$$e^{z_m \tilde{\beta}_m + \tilde{\gamma}_m} \geq 1 \iff z_m \tilde{\beta}_m \geq -\tilde{\gamma}_m. \tag{30}$$

Therefore, the necessary and sufficient condition for convexity is

$$z_m(\beta_{m0} - \beta_{m1}) \geq \gamma_{m1} - \gamma_{m0}. \tag{31}$$

The objective function $f(Z)$ in (21) is convex with respect to z_m under the above constraint because a non-negative weighted sum of convex functions is convex. Moreover, the set of constraints (31) forms a convex set since it is affine in z_m. Consequently, we have a convex program and the proof is complete. ■

Similar argument holds for the general case of $J > 2$, but the convexity constraint becomes cluttered in form and results in loss of brevity. Furthermore, increasing J monotonically decreases the size of the optimization domain, which may raise concerns in terms of practicality.

C. Gradient of Objective Function

Now that we have established conditions under which the above optimization problem becomes a convex optimization problem, we derive the gradient of the objective function so that efficient first-order oracle-based algorithms can be applied to find solutions.

Proposition 3: For $Z \in \mathbb{R}^{T \times L}$, the gradient of $f(Z)$ is

$$\frac{\partial f(Z)}{\partial Z} = \left[\frac{\partial f(Z)}{\partial z_0}, \frac{\partial f(Z)}{\partial z_1}, \cdots, \frac{\partial f(Z)}{\partial z_{T-1}} \right]^\top, \tag{32}$$

where

$$\frac{\partial f(Z)}{\partial z_m} = -\left(\sum_{i=0}^{T-m-1} a^i \right) b(m) \frac{\tilde{\beta}_m e^{z_m \tilde{\beta}_m + \tilde{\gamma}_m}}{\left(1+e^{z_m \tilde{\beta}_m + \tilde{\gamma}_m}\right)^2}, \tag{33}$$

where $\tilde{\beta}_m$ and $\tilde{\gamma}_m$ were defined above for every $m = 0, \ldots, T-1$.

Proof: We show by induction that the objective function can be written

$$f(Z) = \sum_{m=0}^{T-1} \left(\sum_{i=0}^{T-m-1} a^i \right) b(m) \bar{u}(m). \tag{34}$$

For the base cases, let $T = 1$ and $T = 2$. Note that for $T = 1$,

$$f(Z) = b(0)\bar{u}(0) = \sum_{m=0}^{0} \left(\sum_{i=0}^{0} a^i \right) b(m)\bar{u}(m), \tag{35}$$

and for $T = 2$,

$$\begin{aligned} f(Z) &= b(0)\bar{u}(0) + ab(0)\bar{u}(0) + b(1)\bar{u}(1) \\ &= (a^0 + a^1)(b(0)\bar{u}(0)) + a^0 (b(1)\bar{u}(1)) \\ &= \sum_{m=0}^{1} \left(\sum_{i=0}^{1-m} a^i \right) b(m)\bar{u}(m), \end{aligned} \tag{36}$$

where $\bar{u}(m)$ is defined as above for all $m = 0, \ldots, T-1$. As the induction hypothesis, assume that for $T = M$,

$$f(Z) = \sum_{m=0}^{M-1} \left(\sum_{i=0}^{M-m-1} a^i \right) b(m)\bar{u}(m) \tag{37}$$

holds true. Then, for $T = M + 1$ we have

$$f(Z) = \sum_{m=0}^{M} \left(\sum_{i=0}^{M-m} a^i \right) b(m)\bar{u}(m), \tag{38}$$

and we are done. Because $f(Z)$ is dependent on z_m only through $\bar{u}(m)$, we can simply ignore all other terms that do not have z_m and substitute the expression we obtained for $\partial \bar{u}(m)/\partial z_m$ in (27) into our expression for $f(Z)$, which yields $\partial f(Z)/\partial z_m$ in (33) for every $m = 0, \ldots, T-1$. This concludes the proof. ■

D. Projected Gradient Descent Algorithm

Observe that with the aforementioned constrained domain, the problem becomes a convex optimization problem. We can apply the projected gradient descent [20] algorithm to obtain the global minimum. The projected gradient descent has updates of the form

$$z^{\ell+1} = \Pi_{\mathcal{D}} \left(z^\ell - \alpha \frac{\partial f}{\partial Z}(z^\ell) \right), \tag{39}$$

where $\mathcal{D} \doteq \{z_k(\beta_{k0} - \beta_{k1}) \geq \gamma_{k1} - \gamma_{k0}\}$ is a closed, convex subset of $\mathbb{R}^T$, α is the step-size carefully chosen, $\partial f/\partial Z(z^\ell)$ is the gradient derived in Proposition 3, and $\Pi_{\mathcal{D}}$ is the Euclidean projection operator defined as

$$\Pi_{\mathcal{D}}(y) \doteq \arg\min_{x \in \mathcal{D}} \|x - y\|_2^2. \tag{40}$$

The convergence rate of this algorithm depends on the complexity of the projection operator, which we do not investigate further in this paper and is detailed in textbooks such as [19].

IV. Application to Demand Response

In this section, we apply the above convex optimization framework of DSDCM to demand response (DR). In DR, an electric power system operator requests a reduced level of power consumption from specific DR participants. Examples of DR participants include residential air conditioners, freezers in grocery store distribution centers, or electric vehicle charging stations. In many cases, the power system operator is not able to directly command a power reduction from the loads. Instead, the operator can indirectly adjust the probability of participation by providing (typically economic) incentives. This application properly fits our framework, where we seek to control the system trajectory where system inputs are randomly generated from human choices, but conditioned on incentive signals that we can provide.

A. System Dynamics and Discrete Choices

From (1), we extend the system to consider N human actuators, or participants, in a DR contract, uniquely defined by DCM. We then simplify the system by considering only the total non-complied power load over all participants at time-step k, which is represented by a one-dimensional state $x(k) \in \mathbb{R}$. The system is further simplified by defining the system dynamics as the cumulative sum of the system inputs, i.e. $A = a = 1$. We finally take the expectation of the system dynamics to fit the aforementioned optimization framework. The state mean dynamics is then captured by

$$\bar{x}(k+1) = \bar{x}(k) + B(k)^\top \bar{u}(k), \tag{41}$$

where $B(k)$ is the vector of reducible power loads over all participants at time-step k. The choice of the n-th participant at time-step k is binary; it is either compliance ($u_n(k) = 0$) or non-compliance ($u_n(k) = 1$). We assume knowledge of their baseline power consumption behavior and reducible power over the operating hours. It is also assumed that the parameters $\beta_{m0}^{(n)}, \beta_{m1}^{(n)}, \gamma_{m0}^{(n)}, \gamma_{m1}^{(n)}$ of the utility functions for compliance and non-compliance are known and fixed for all participants.

B. Formulation of the Optimization Problem

The objective of the service operator is to minimize non-complied power loads while also minimizing price compensations by determining a sequence of price compensations z_m for the participants. We only consider one controllable attribute at every time-step k, i.e. $Z \in \mathbb{R}^T$. The optimization problem is then formulated as

$$\min_{Z\in\mathbb{R}^T} \quad \sum_{k=1}^{T}\left(\sum_{m=0}^{k-1} B(m)^\top \bar{u}(m)\right) + \lambda\|Z\|_2 \tag{42}$$

$$\text{s.t.} \quad x(0) = x_0 = 0$$

$$B(m) = [b_1(m)\ b_2(m)\ \cdots\ b_N(m)]^\top \in \mathbb{R}^{N\times 1} \tag{43}$$

$$\bar{u}(m) = [\bar{u}_1(m), \bar{u}_2(m), \cdots, \bar{u}_N(m)]^\top \in \mathbb{R}^{N\times 1} \tag{44}$$

$$\bar{u}_n(m) = \frac{e^{\beta_{m1}^{(n)} z_m + \gamma_{m1}^{(n)}}}{e^{\beta_{m0}^{(n)} z_m + \gamma_{m0}^{(n)}} + e^{\beta_{m1}^{(n)} z_m + \gamma_{m1}^{(n)}}} \tag{45}$$

$$z_m(\beta_{m0}^{(n)} - \beta_{m1}^{(n)}) \geq \gamma_{m1}^{(n)} - \gamma_{m0}^{(n)}, \tag{46}$$

where $Z = [z_0, z_1, \cdots, z_{T-1}]^\top$, $b_n(m)$ is the non-complied power load of the n-th participant at time-step m, and λ is the regularization parameter penalizing price compensations–i.e. control effort.

The first term in the objective function (42) denotes the expected sum of non-complied power loads and the second term denotes the penalty of price compensations. Equation (43) is the vector of non-complied power loads over N participants at time-step m. Equation (44) represents the vector of expected decisions of N participants at time-step m and (45) denotes the expected decision value of the n-th participant at time-step m. Parameter $\gamma_{mi}^{(n)}$, $i \in \{0, 1\}$, is an exogenous uncontrolled variable, such as temperature. Equation (46) is the inequality constraint that retains the convexity of the optimization problem, as derived in Section III-B. Note that the expected sum of states is convex in the domain of z_m that satisfies the constraints (c.f. III-B) and that the L_2 regularization norm is convex. The sum of two non-negative convex functions is still convex, and thus we have a convex optimization problem.

C. Simulation

We simulate a simple instance of the above case study. We consider totals of 5 DR participants ($N = 5$) and 10 operating hours ($T = 10$). In the simulation, the optimization problem was solved 100 times. Denote by Z^* the solution to the above optimization problem. At each trial, the parameters in the discrete choice models of DR participants were fixed and known, and exogenous attributes of DCM were sampled from the standard normal distribution. Fig. 2 illustrates that the optimal power compensations and the average reducible power by DR over all participants are strongly correlated, as expected. This result implies that accurate prediction of reducible power is important in practice.

Fig. 3 implies that the expected non-complied power load sum and price compensation are negatively correlated. The correlation is linear; linear regression yields R^2 value of 0.997. Z^* is the solution to the optimization problem and $\bar{u}^*(m)$ is $\bar{u}(m)$ evaluated with $z_m = Z^*$.

D. Discussion

The formulated optimization problem can be extended in several practically useful ways while retaining convexity. For example, one can incorporate uncertainties in non-complied power loads (43) and exogenous variables in DCM (45). This implies that the above optimization problem can be integrated with power load forecasting methods [21], [22] and also exogenous variable forecasting, e.g. temperatures [23]. These applications remain open for future work.

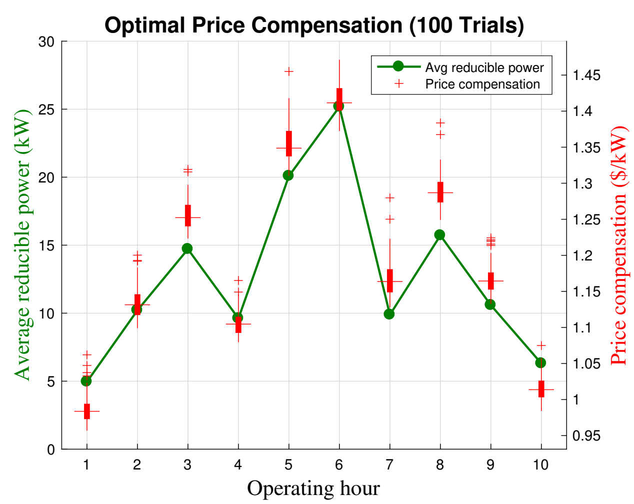

Fig. 2. Simulation result for the case study where the total operating hours $T = 10$ and total participants $N = 5$. Exogenous attributes $\gamma_{mi}^{(n)}$, $i \in \{0, 1\}$, were sampled randomly from the standard normal distribution.

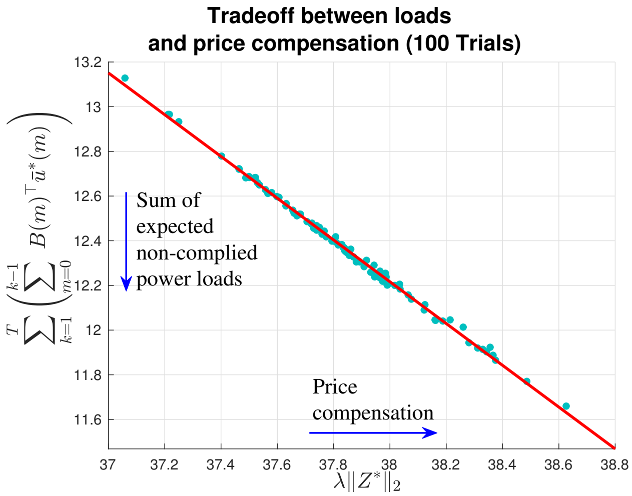

Fig. 3. Trade-off between the sum of expected non-complied power load and price compensation. The compensation penalty parameter was set to $\lambda = 10$. The sum of expected non-complied power load and price compensation are negatively and linearly correlated.

There are several limitations to the proposed optimization problem. The convexity constraints (46) confine the optimal solutions to reside within the constrained domain. The constrained domain may yield suboptimal solutions with respect to the original unconstrained optimization problem. That said, the convex optimization problem can be easily solved. Therefore, retaining convexity by constraining the domain, for many applications, is reasonable and acceptable. Another limitation is that collecting reasonable amount of data to model human behavior can be quite difficult. Consequently, using DCM in the suggested framework could be disputable in application settings where human decisions are hardly observable. Nevertheless, this paper demonstrates how data can be utilized, if available, which provides intuition on how to address human behavior in dynamical systems.

V. Conclusion

This paper provided a first investigation of dynamical systems with human actuation, where system inputs are generated according to Discrete Choice Models (DCM). We referred to such system as a Dynamic System with Discrete Choice Models (DSDCM), defined this class of systems, and provided mathematical analyses of the system equilibrium, stability, and controllability. We also proposed an optimal control framework of DSDCM that could be formulated as a convex program. Finally, we applied the above convex optimization problem to the demand response (DR) problem, where the probability of DR participation was determined by DCM. Future work consists of developing a control scheme for DSDCM under uncertainties, and extending to modeling human actuated system in stochastic hybrid system framework.

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