Robust Fault Diagnosis of Uncertain One-dimensional Wave Equations

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***Abstract*— Unlike its Ordinary Differential Equation (ODE) counterpart, fault diagnosis of Partial Differential Equations (PDE) has received limited attention in existing literature. The main difficulty in PDE fault diagnosis arises from the spatio-temporal evolution of the faults, as opposed to temporal-only fault dynamics in ODE systems. In this work, we develop a fault diagnosis scheme for one-dimensional wave equations. A key aspect of this fault diagnosis scheme is to distinguish the effect of uncertainties from faults. The scheme consists of a PDE observer whose output error is treated as a fault indicating residual signal. Furthermore, a threshold on the residual signal is utilized to infer fault occurrence. The convergence properties of the PDE observer and residual signal are analyzed via Lyapunov stability theory. The threshold is designed based on the uncertain residual dynamics and the upper bound of the uncertainties. Simulation studies are performed to illustrate the effectiveness of the proposed fault diagnosis scheme.**

I. INTRODUCTION

Wave equations are typically used in science and engineering to model phenomena such as sound, light, pressure, water waves, and have extensive applications in fluid dynamics, electromagnetics and acoustics [1]. A rich body of literature exists on control problems for wave equations [2], [3], [4], [5], [6], [7]. Estimation problems have also received significant attention in [8], [9], [10], [11], [12]. Unlike control and estimation problems, fault diagnosis of wave equations has not received its due attention. Namely, fault diagnosis can be critically important for maintaining safety in wave-type systems/processes. For example, a fault diagnosis scheme would be able to mitigate the possibility of oil spill by significantly improving pipeline protection [13] and blackouts via power systems monitoring [14].

Generally speaking, fault diagnosis of PDEs has received significantly less attention than ODEs. The typical approach for PDE fault diagnosis is *early lumping*, where the design is done on an ODE approximation of the original PDE [15] [16] [17]. Following this *early lumping* approach, a Kalman filtering based scheme is proposed in [18] for fault diagnosis of wave PDEs. However, such finite dimensional approximations sometimes neglect the higher order but important modes of the system, which in turn may lead to control/observation spillover [19]. Furthermore, faults may cause significant changes in the original PDE model, further decreasing the accuracy of the approximated model. Apart from *early lumping* approaches, some PDE fault diagnosis schemes utilize operator theory to design infinite dimensional observers [20] [21]. Similar schemes have been designed for wave equations in [22]. Although these schemes overcome many issues of *early lumping*, an operator theory-based design leads to high computational complexity in implementation and consequently has translated to very few engineering applications.

In this work, we propose a fault diagnosis scheme for uncertain wave PDEs that utilizes a PDE backstepping transformation in conjunction with Lyapunov stability analysis [23]. Specifically, the proposed scheme does not resort to any finite dimensional approximations or operator theory in its design, unlike the aforementioned approaches. In our problem setting, we assume the presence of an in-domain distributed uncertainty along with boundary state measurement. The scheme consists of a boundary error injection-based PDE observer. The boundary error is treated as a *residual signal* which acts as a fault indicator. Furthermore, we design a constant *threshold* based on the observer error dynamics and uncertainty bound. The *residual signal* is evaluated against the *threshold* to infer fault occurrence. This *threshold*-based technique provides robustness to the uncertainty, which is an important feature of the proposed scheme. The rest of the paper is structured as follows. Section II describes the problem formulation. Section III details the algorithm design. Section IV illustrates the algorithm with simulation studies and Section V concludes the work.

Throughout the paper, we have used the following notation: $\|u(x,t)\| = \sqrt{\int_0^1 u^2(x,t)dx}$, $u_t = \frac{\partial u}{\partial t}, u_{tt} = \frac{\partial^2 u}{\partial t^2}, u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$. For $a, b, \lambda \in \mathbb{R}$ with $\lambda > 0$, the following inequalities hold:

$$ab \leq \frac{\lambda}{2}a^2 + \frac{1}{2\lambda}b^2, \quad ab \geq -\frac{\lambda}{2}a^2 - \frac{1}{2\lambda}b^2. \tag{1}$$

The Cauchy-Schwarz inequality is given by the following

$$\int_0^1 f_1(x,t)f_2(x,t)dx \leq \|f_1\| \, \|f_2\| \, . \tag{2}$$

II. PROBLEM FORMULATION

Consider a class of wave equations represented by the following PDE:

$$u_{tt}(x,t) = u_{xx}(x,t) + \Delta(x,t) + \psi(x,t), \tag{3}$$

with the boundary and initial conditions

$$u_x(0,t) = -Ku_t(0,t), u_x(1,t) = Q(t), u(x,0) = u_0(x), \tag{4}$$

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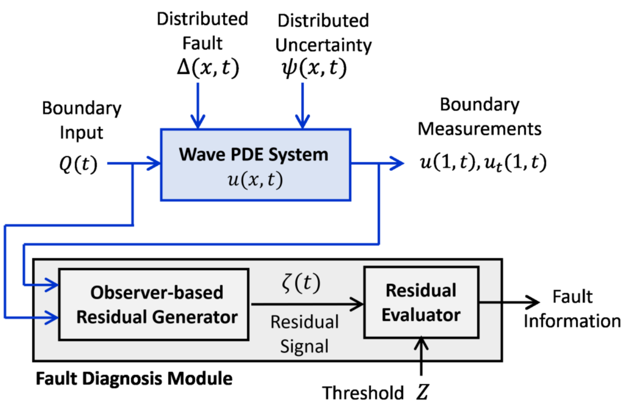

Fig. 1. Fault diagnosis scheme.

where $t \in [0, \infty)$ represents time and $x \in [0, 1]$ is the spatial coordinate. Source term $\psi(x, t)$ represents an unknown spatially distributed uncertainty, $\Delta(x, t)$ is an unknown spatially distributed fault, $K \in \mathbb{R}^+$ is a known positive constant, and $Q(t) \in \mathbb{R}$ is a known boundary input signal.

Assumption 1. *The following boundary measurements are available: $u(1, t)$ and $u_t(1, t)$. Furthermore, the uncertainty $\psi(x, t)$ and the fault $\Delta(x, t)$ are bounded in the sense of spatial $\mathcal{L}_2$ norm: $\|\psi(x, t)\| \leq \overline{\psi}$, $\|\Delta(x, t)\| \leq \overline{\Delta}$.*

The goal is to detect the occurrence of the fault $\Delta(x, t)$ in the presence of the uncertainty $\psi(x, t)$. To achieve this goal, we propose the diagnostic scheme depicted in Fig. 1. The scheme works in a cascaded manner: First, a residual signal $\zeta(t)$ is generated by an output-injection based observer. Next, the residual signal $\zeta(t)$ is compared against a threshold Z. Finally, we use the following logic to detect the occurrence of the fault $\Delta(x, t)$: $\zeta(t) > Z \quad \rightarrow \quad$ fault occurred, $\zeta(t) \leq Z \quad \rightarrow \quad$ no fault occurred. In the following section, we detail the design of the observer and the threshold.

III. DESIGN AND ANALYSIS OF THE DIAGNOSTIC SCHEME

We design the following output injection-based observer to generate the residual signal $\zeta(t)$:

$$\hat{u}_{tt}(x, t) = \hat{u}_{xx}(x, t) + K_1 \tilde{u}(1, t) + K_2 \tilde{u}_t(1, t), \tag{5}$$

$$\hat{u}_x(0, t) = -K \hat{u}_t(0, t) + K_3 \tilde{u}(1, t) + K_4 \tilde{u}_t(1, t), \tag{6}$$

$$\hat{u}_x(1, t) = Q(t) + K_5 \tilde{u}(1, t) + K_6 \tilde{u}_t(1, t), \tag{7}$$

$$\zeta(t) = \tilde{u}(1, t), \tag{8}$$

with the initial condition

$$\hat{u}(x, 0) = \hat{u}_0(x), \tag{9}$$

where $\tilde{u}(1, t) = u(1, t) - \hat{u}(1, t)$, and $K_i \in \mathbb{R}, i \in \{1, 2, .., 6\}$ are the observer gains to be designed. Subtracting (5)-(7) from (3)-(4) yields the observer error dynamics:

$$\tilde{u}_{tt}(x, t) = \tilde{u}_{xx}(x, t) + \Delta(x, t) + \psi(x, t) - K_1 \tilde{u}(1, t) - K_2 \tilde{u}_t(1, t), \tag{10}$$

$$\tilde{u}_x(0, t) = -K \tilde{u}_t(0, t) - K_3 \tilde{u}(1, t) - K_4 \tilde{u}_t(1, t), \tag{11}$$

$$\tilde{u}_x(1, t) = -K_5 \tilde{u}(1, t) - K_6 \tilde{u}_t(1, t). \tag{12}$$

Next, we use the following backstepping transformation $\tilde{u}(x, t) \mapsto v(x, t)$

$$\tilde{u}(x, t) = v(x, t) + P \int_x^1 v_t(y, t) dy, \tag{13}$$

which transforms the error dynamics (10)-(12) into the following target error dynamics

$$v_{tt}(x, t) = v_{xx}(x, t) + g(x, t) + h(x, t), \tag{14}$$

$$v_x(0, t) = c_1 v_t(0, t) + g_2(t) + h_2(t), \tag{15}$$

$$v_x(1, t) = -c_2 v(1, t). \tag{16}$$

where P is a transformation constant to be determined, and $c_1, c_2 \in \mathbb{R}^+$ are design parameters. Note that a constant transformation gain P is sufficient in this case, instead of a gain kernel [9]. The terms $g(x, t)$ and $h(x, t)$ are the transformed fault and uncertainty in the v-domain, given by

$$\Delta(x, t) = g(x, t) + P \int_x^1 g_t(y, t) dy, \tag{17}$$

$$\psi(x, t) = h(x, t) + P \int_x^1 h_t(y, t) dy. \tag{18}$$

Furthermore, the terms $g_2(t)$ and $h_2(t)$ are defined as follows

$$g_2(t) = \frac{KP}{KP - 1} \int_0^1 g(y, t) dy, \tag{19}$$

$$h_2(t) = \frac{KP}{KP - 1} \int_0^1 h(y, t) dy. \tag{20}$$

Next, we find the conditions on P and $K_i, i \in \{1, 2, .., 6\}$ for which the transformation (13) exists.

Conditions on P and K_i: Differentiating (13) twice with respect to t and considering $v_{xt}(1, t) = -c_2 v_t(1, t) = -c_2 \tilde{u}_t(1, t)$, we have

$$\tilde{u}_{tt}(x, t) = v_{tt}(x, t) + P v_{xt}(1, t) - P v_{xt}(x, t) + P \int_x^1 g_t(y, t) dy + P \int_x^1 h_t(y, t) dy. \tag{21}$$

Differentiating (13) twice with respect to x, we have

$$\tilde{u}_{xx}(x, t) = v_{xx}(x, t) - P v_{xt}(x, t). \tag{22}$$

Next, subtracting (21) from (22) and comparing both sides, we get

$$K_1 = P c_2, \qquad K_2 = 0. \tag{23}$$

Differentiating (13) once with respect to x, we have

$$\tilde{u}_x(x, t) = v_x(x, t) - P v_t(x, t). \tag{24}$$

Now, substituting $x = 1$ in (24), and considering (16) and $v_t(1, t) = \tilde{u}(1, t)$ we have

$$\tilde{u}_x(1, t) = -c_2 \tilde{u}(1, t) - P \tilde{u}_t(1, t). \tag{25}$$

Comparing (25) with (7), we get

$$K_5 = c_2, \qquad K_6 = P. \tag{26}$$

From the transformation (13), we also get

$$\tilde{u}_x(0,t) = (c_1 - P)v_t(0,t) + g_2(t) + h_2(t). \quad (27)$$

and

$$\tilde{u}_t(0,t) = v_t(0,t) + Pv_x(1,t) - Pv_x(0,t) + P\int_0^1 g(y,t)dy + +P\int_0^1 h(y,t)dy. \quad (28)$$

Now, from (27) and (28), we can get

$$\begin{aligned}\tilde{u}_x(0,t) + K\tilde{u}_t(0,t) &= (c_1 - P)v_t(0,t) + g_2(t) + h_2(t) \\ &+ Kv_t(0,t) + KPv_x(1,t) - KPv_x(0,t) \\ &+ KP\int_0^1 g(y,t)dy + KP\int_0^1 h(y,t)dy, \quad (29)\end{aligned}$$

which can be further simplified as

$$\tilde{u}_x(0,t) + K\tilde{u}_t(0,t) = (c_1 - P + K - KPc_1)v_t(0,t) - KPc_2\tilde{u}(1,t). \quad (30)$$

Comparing (30) with (11), we obtain

$$K_3 = KPc_2, \qquad K_4 = 0, \qquad P = \frac{c_1 + K}{1 + Kc_1}. \quad (31)$$

Finally, the transformation constant and the observer gains are summarized as

$$P = \frac{c_1 + K}{1 + Kc_1}, \quad K_1 = Pc_2, \quad K_2 = 0, \quad (32)$$

$$K_3 = KPc_2, \quad K_4 = 0, \quad K_5 = c_2, \quad K_6 = P. \quad (33)$$

Remark 1. *The fault $\Delta(x,t)$ and the uncertainty $\psi(x,t)$ go through the same transformation as the error variable $\tilde{u}(x,t)$ (as evident from (17)-(18)). Due to Assumption 1 and invertibility of the transformation, the fault and uncertainties are also bounded in the v-domain as:*

$$\|g\| \le \overline{g}, \quad \|h\| \le \overline{h}, \quad |g_2| \le \overline{g}_2, \quad |h_2| \le \overline{h}_2, \quad (34)$$

Theorem 1. *Consider the target error dynamics (14)-(16) and the bounds in (34). If*

(i) Assumption 1 holds, and
(ii) $c_1 > 0, c_2 > 1$, and
(iii) c_1 satisfies the condition $(-c_1 + \epsilon + \epsilon c_1^2 + \frac{|2\epsilon c_1 - 1|}{2}) < 0$ with $\epsilon \in (0, \frac{1}{2})$ being a small positive number

then the following are true:

(i) in the presence of no uncertainty and no fault, i.e. $g(x,t) = 0$ and $h(x,t) = 0$ (equivalently, $\psi(x,t) = 0$ and $\Delta(x,t) = 0$), the residual signal $\zeta(t)$ defined in (8) will achieve exponential convergence to zero.
(ii) in the presence of uncertainty and/or fault, i.e. $g(x,t) \neq 0$ and/or $h(x,t) \neq 0$ (equivalently, $\psi(x,t) \neq 0$ and/or $\Delta(x,t) \neq 0$), the residual signal $\zeta(t)$ will be upper bounded as $t \to \infty$, by the following bound:

$$|\zeta(t)| \le \overline{Z} = \sqrt{\frac{2}{c_2}\sqrt{\frac{\overline{\beta} + \sqrt{\overline{\beta}^2 + 4\alpha\overline{\gamma}}}{2\alpha}}}. \quad (35)$$

where $\alpha = \frac{\epsilon}{4}, \overline{\beta} = \overline{g} + \overline{h}$ and $\overline{\gamma} = \left(\epsilon + \frac{|2\epsilon c_1 - 1|}{2}\right)(\overline{g}_2 + \overline{h}_2)^2$.

Proof: Consider the Lyapunov functional candidate

$$W(t) = \frac{c_2}{2}v^2(1,t) + \frac{1}{2}\int_0^1 v_x^2(x,t)dx + \frac{1}{2}\int_0^1 v_t^2(x,t)dx + \epsilon\int_0^1 (x-2)v_x(x,t)v_t(x,t)dx. \quad (36)$$

where $\epsilon > 0$ is a small positive number. Next, we prove the positive definiteness of $W(t)$.

Positive Definiteness of $W(t)$: Considering the term inside the integral of the fourth term on the right hand side of (36) and applying second inequality in (1), we have

$$(x-2)v_xv_t \ge -\frac{\lambda}{2}(x-2)^2v_x^2 - \frac{1}{2\lambda}v_t^2 \quad (37)$$

$$\ge -2\lambda v_x^2 - \frac{1}{2\lambda}v_t^2 \quad (38)$$

$$= -v_x^2 - v_t^2 \quad (\text{for} \quad \lambda = \frac{1}{2}). \quad (39)$$

Using (39), the lower bound of (36) can be written as

$$W(t) \ge \frac{c_2}{2}v^2(1,t) + \left(\frac{1}{2} - \epsilon\right)\left(\int_0^1 v_x^2(x,t)dx + \int_0^1 v_t^2(x,t)dx\right) \quad (40)$$

Hence $W(t)$ is a positive definite Lyapunov functional for $\epsilon < \frac{1}{2}$.

Negative Definiteness of $\dot{W}(t)$: Next, we explore the conditions on the negative definiteness of $\dot{W}(t)$ which can be written as

$$\begin{aligned}\dot{W}(t) &= c_2v(1)v_t(1) + \int_0^1 v_xv_{xt}dx + \int_0^1 v_tv_{tt}dx \\ &+ \epsilon\int_0^1 (x-2)v_{xt}v_tdx + \epsilon\int_0^1 (x-2)v_xv_{xx}dx \\ &+ \epsilon\int_0^1 (x-2)v_x(g+h)dx. \quad (41)\end{aligned}$$

Consider the third term on the right hand side of (41). Applying integration by parts and utilizing inequality (2), the upper bound of this term can be written as

$$\begin{aligned}\int_0^1 v_tv_{tt}dx &= \int_0^1 v_tv_{xx}dx + \int_0^1 v_t(g+h)dx. \\ &\le -c_2v_t(1)v(1) - c_1v_t(0)^2 - v_t(0)(g_2 + h_2) \\ &- \int_0^1 v_xv_{xt}dx + \|v_t\|\,\|g+h\| \quad (42)\end{aligned}$$

Next, consider the fourth term on the right hand side of (41). Applying integration by parts, we can write

$$\begin{aligned}&\epsilon\int_0^1 (x-2)v_{xt}v_tdx \\ &= -\epsilon v_t(1)^2 + 2\epsilon v_t(0)^2 - \epsilon\int_0^1 (x-2)v_{xt}v_tdx - \epsilon\int_0^1 v_t^2dx. \quad (43)\end{aligned}$$

From (43), we can write

$$\epsilon\int_0^1 (x-2)v_{xt}v_t dx = -\frac{\epsilon}{2}v_t(1)^2 + \epsilon v_t(0)^2 - \frac{\epsilon}{2}\int_0^1 v_t^2 dx. \tag{44}$$

Next, consider the fifth term on the right hand side of (41). Applying integration by parts, we can write

$$\epsilon\int_0^1 (x-2)v_x v_{xx} dx = -\epsilon v_x(1)^2 + 2\epsilon v_x(0)^2 - \epsilon\int_0^1 (x-2)v_{xx}v_x dx - \epsilon\int_0^1 v_x^2 dx. \tag{45}$$

From (45), we have

$$\epsilon\int_0^1 (x-2)v_x v_{xx} dx = -\frac{\epsilon}{2}v_x(1)^2 + \epsilon v_x(0)^2 - \frac{\epsilon}{2}\int_0^1 v_x^2 dx. \tag{46}$$

Applying the boundary conditions (15)-(16), we can re-write (46) as

$$\epsilon\int_0^1 (x-2)v_x v_{xx} dx = -\frac{\epsilon}{2}c_2^2 v(1)^2 + \epsilon c_1^2 v_t(0)^2 + \epsilon(g_2+h_2)^2 + 2\epsilon c_1 v_t(0)(g_2+h_2) - \frac{\epsilon}{2}\int_0^1 v_x^2 dx. \tag{47}$$

Considering (42), (44) and (47), we can write the upper bound of $\dot{W}(t)$ as:

$$\begin{aligned}\dot{W}(t) \leq &\Big(-c_1+\epsilon+\epsilon c_1^2\Big)v_t(0)^2 + \Big(2\epsilon c_1 - 1\Big)v_t(0)(g_2+h_2) + \|v_t\|\,\|g+h\| - \frac{\epsilon}{2}v_t(1)^2 - \frac{\epsilon}{2}\|v_t\|^2 \\ &- \frac{\epsilon}{2}c_2^2 v(1)^2 + \epsilon(g_2+h_2)^2 - \frac{\epsilon}{2}\|v_x\|^2 + 2\epsilon\|v_x\|\,\|g+h\|\end{aligned} \tag{48}$$

Applying the inequality $ab \leq |a|\,|b|$ on (48), we can further write

$$\begin{aligned}\dot{W}(t) \leq &\Big(-c_1+\epsilon+\epsilon c_1^2\Big)v_t(0)^2 + |2\epsilon c_1 - 1|\,|v_t(0)|\,|g_2+h_2| + \|v_t\|\,\|g+h\| - \frac{\epsilon}{2}v_t(1)^2 \\ &- \frac{\epsilon}{2}\|v_t\|^2 - \frac{\epsilon}{2}c_2^2 v(1)^2 + \epsilon(g_2+h_2)^2 - \frac{\epsilon}{2}\|v_x\|^2 + 2\epsilon\|v_x\|\,\|g+h\|\end{aligned} \tag{49}$$

Using the first inequality in (1) with $\lambda = 1$, we can write

$$|v_t(0)|\,|(g_2+h_2)| \leq \frac{1}{2}|v_t(0)|^2 + \frac{1}{2}|(g_2+h_2)|^2 \tag{50}$$

Utilizing (50), we can re-write (49) as

$$\begin{aligned}\dot{W}(t) \leq &-\frac{\epsilon}{2}c_2^2 v(1)^2 - \frac{\epsilon}{2}\|v_x\|^2 - \frac{\epsilon}{2}\|v_t\|^2 + \Big(-c_1+\epsilon+\epsilon c_1^2 + \frac{|2\epsilon c_1 - 1|}{2}\Big)|v_t(0)|^2 + \|v_t\|\,\|g+h\| \\ &+ \Big(\epsilon + \frac{|2\epsilon c_1 - 1|}{2}\Big)(g_2+h_2)^2 + 2\epsilon\|v_x\|\,\|g+h\|\end{aligned} \tag{51}$$

For a small ϵ, we have $\Big(-c_1+\epsilon+\epsilon c_1^2 + \frac{|2\epsilon c_1 - 1|}{2}\Big) < 0$. Hence, (51) can be re-written as

$$\dot{W}(t) \leq W_1 + W_2 + W_3, \tag{52}$$

where
$$W_1 = -\frac{\epsilon}{2}c_2^2 v(1)^2 - \frac{\epsilon}{2}\|v_x\|^2 - \frac{\epsilon}{2}\|v_t\|^2, \tag{53}$$

$$W_2 = \|v_t\|\,\|g+h\| + 2\epsilon\|v_x\|\,\|g+h\|, \tag{54}$$

$$W_3 = \Big(\epsilon + \frac{|2\epsilon c_1 - 1|}{2}\Big)(g_2+h_2)^2, \tag{55}$$

Now, applying first inequality in (1) with $\lambda = \frac{1}{2}$, we can write:

$$(x-2)v_x v_t \leq v_x^2 + v_t^2 \tag{56}$$

and hence

$$-\frac{\epsilon}{4}\int_0^1 (x-2)v_x v_t dx \geq -\frac{\epsilon}{4}\int_0^1 v_x^2 dx - \frac{\epsilon}{4}\int_0^1 v_t^2 dx \tag{57}$$

First, we will find the upper bound of W_1. Considering (57) and choosing $c_2 > 1$, we can upper bound W_1 in (53) as

$$\begin{aligned}W_1 &= -\frac{\epsilon}{2}c_2^2 v(1)^2 - \frac{\epsilon}{2}\|v_x\|^2 - \frac{\epsilon}{2}\|v_t\|^2 \\ &\leq -\frac{\epsilon}{4}c_2 v(1)^2 - \frac{\epsilon}{4}\|v_x\|^2 - \frac{\epsilon}{4}\|v_t\|^2 - \frac{\epsilon}{4}\int_0^1 (x-2)v_x v_t dx \\ &\leq -\frac{\epsilon}{8}c_2 v(1)^2 - \frac{\epsilon}{8}\|v_x\|^2 - \frac{\epsilon}{8}\|v_t\|^2 - \frac{\epsilon}{4}\int_0^1 (x-2)v_x v_t dx \\ &= -\frac{\epsilon}{4}W(t)\end{aligned} \tag{58}$$

Next, we will find the upper bound of W_2.

$$\begin{aligned}W_2 &= \|v_t\|\,\|g+h\| + 2\epsilon\|v_x\|\,\|g+h\| \\ &\leq \max\{\|v_t\|, 2\epsilon\|v_x\|\}\,\|g+h\|\end{aligned} \tag{59}$$

If $\max\{\|v_t\|, 2\epsilon\|v_x\|\} = \|v_t\|$, then we can write

$$W_2 \leq \|v_t\|\,\|g+h\| \leq \sqrt{W}\,\|g+h\| \tag{60}$$

as $\|v_t\| \leq \sqrt{W}$. Using similar argument, in case of $\max\{\|v_t\|, 2\epsilon\|v_x\|\} = 2\epsilon\|v_x\|$, we have $W_2 \leq \sqrt{W}\,\|g+h\|$. Hence, in either case, we can write

$$W_2 \leq \sqrt{W}\,\|g+h\| \tag{61}$$

Finally, using (58) and (61), we can re-write (52) as

$$\dot{W} \leq -\alpha W + \beta\sqrt{W} + \gamma \tag{62}$$

where $\alpha = \frac{\epsilon}{4} > 0$,$\beta = \|g+h\| > 0$ and $\gamma = W_3 > 0$. Next, consider the following two cases:

Case 1: When there is no fault nor uncertainty: For no fault and no uncertainty, we have $g, h = 0$ and hence, $g_2, h_2 = 0$, which makes $\beta, \gamma = 0$. Under this condition, the solution to the differential inequality can be written as:

$$W(t) \leq e^{-\alpha t}W(0). \tag{63}$$

Consider the term $W(0) = \frac{c_2}{2}v^2(1,0) + m_1 + m_2 + m_3$, where $m_1 = \frac{1}{2}\int_0^1 v_x^2(x,0)dx, m_2 = \frac{1}{2}\int_0^1 v_t^2(x,0)dx, m_3 = \epsilon\int_0^1 (x-2)v_x(x,0)v_t(x,0)dx$ are finite constants (as the initial conditions are assumed to be finite). We can always find a finite constant D for which the following is true:

$W(0) \leq Dv^2(1,0)$. Considering this aforementioned argument and the fact that $\frac{c_2}{2}v^2(1,t) \leq W(t)$, we can write

$$v^2(1,t) \leq Me^{-\alpha t}v^2(1,0) \implies v(1,t) \leq \sqrt{M}e^{-\frac{\alpha}{2}t}v(1,0). \tag{64}$$

where $M = \frac{2D}{c_2}$. Hence, we can conclude the exponential convergence of $v(1,t) \to 0$ as $t \to \infty$ in the absence of fault and uncertainty.

Case 2: When there is fault and/or uncertainty: Under this condition, we have $\beta, \gamma \neq 0$. We can re-write (62) as

$$\dot{W} \leq -\alpha W + \overline{\beta}\sqrt{W} + \overline{\gamma} \tag{65}$$

with $\overline{\beta} = \overline{g} + \overline{h}$ and $\overline{\gamma} = \left(\epsilon + \frac{|2\epsilon c_1 - 1|}{2}\right)(\overline{g}_2 + \overline{h}_2)^2$. Hence, negative definiteness of $\dot{W}$ will only be guaranteed under the following condition

$$\alpha W - \overline{\beta}\sqrt{W} - \overline{\gamma} > 0 \tag{66}$$

which is equivalent to $W > \sqrt{\frac{\overline{\beta} + \sqrt{\overline{\beta}^2 + 4\alpha\overline{\gamma}}}{2\alpha}}$. Hence, we can conclude that

$$\lim_{t\to\infty} W(t) \leq \sqrt{\frac{\overline{\beta} + \sqrt{\overline{\beta}^2 + 4\alpha\overline{\gamma}}}{2\alpha}} \tag{67}$$

$$\implies \lim_{t\to\infty} \zeta(t) = v(1,t) \leq \overline{Z} = \sqrt{\frac{2}{c_2}\sqrt{\frac{\overline{\beta} + \sqrt{\overline{\beta}^2 + 4\alpha\overline{\gamma}}}{2\alpha}}} \tag{68}$$

Remark 2. *The convergence of the observer estimation error $\tilde{v}(x,t)$ can be analyzed in terms of the norm $\left(v^2(1,t) + \int_0^1 v_x^2(x,t)dx + \int_0^1 v_t^2(x,t)dx\right)^{\frac{1}{2}}$, following the steps given in [9].*

Remark 3. *The bound Z in Theorem 1 is the upper bound of the residual signal $\zeta(t)$ under fault and uncertainty. In the presence of uncertainty but no fault,i.e. $g(x,t) = 0$ and $h(x,t) \neq 0$, the upper bound of $\zeta(t)$ reduces to*

$$|\zeta(t)| \leq Z = \sqrt{\frac{2}{c_2}\sqrt{\frac{\overline{\beta}_1 + \sqrt{\overline{\beta}_1^2 + 4\alpha\overline{\gamma}_1}}{2\alpha}}}, \tag{69}$$

where $\alpha = \frac{\epsilon}{4}, \overline{\beta}_1 = \overline{h}$ and $\overline{\gamma}_1 = \left(\epsilon + \frac{|2\epsilon c_1 - 1|}{2}\right)\overline{h}_2^2$. We use this upper bound Z as a constant threshold on the residual $\zeta(t)$ with the following fault detection logic: $\zeta(t) > Z \quad \rightarrow$ fault occurred, $\zeta(t) \leq Z \quad \rightarrow \quad$ no fault occurred.

IV. SIMULATION STUDIES

In this section, we present simulation studies to illustrate the performance of the proposed scheme. We have chosen the following parameter and input for our open-loop plant: $K = 300$ and $Q(t) = 100\sin(\omega t)$ with $\omega = 0.003$. The PDE is implemented using the finite difference method with time-step $\delta_T = 0.001s$ and spatial discretization step $\delta_x = 0.01$. The distributed state response $u(x,t)$ under nominal conditions, that is, no uncertainty nor fault is shown in Fig. 2. The PDE observer and the residual are initialized with different initial conditions than the plant. Next, the uncertainty $\psi(x,t) = K_1 u_{xx} + K_2$ with $K_1 = 0.1$ and $K_2 = 10$ is added to the plant dynamics. The choice of ψ is motivated by the potential uncertainty in the wave speed and some additive disturbances. Accordingly, the threshold $(\zeta(t))$ was chosen based on Remark 3 with $\epsilon = 0.005, c_1 = 1.2, c_2 = 1.5$. The evolution of the residual under uncertainty (but no fault) is shown in Fig. 3. As expected, the residual $\zeta(t)$ does not converge to zero but remains bounded within the threshold – indicating no fault. Finally, a fault $g(x,t) = 50(1 - e^{-0.003(t-10)})\sin(x)$ (see Fig. 4) was injected at $t = 10s$ in the uncertain plant. The evolution of the residual $\zeta(t)$ under the uncertainty and the fault is shown in Fig. 5. As expected, 1.8 sec after the fault injection, the residual $\zeta(t)$ crosses the threshold Z indicating a fault occurrence.

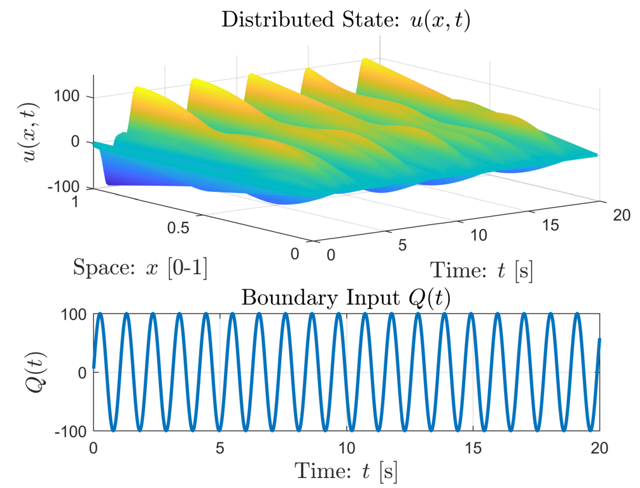

Fig. 2. Distributed state response of the open-loop plant under nominal condition (without any fault or any uncertainty).

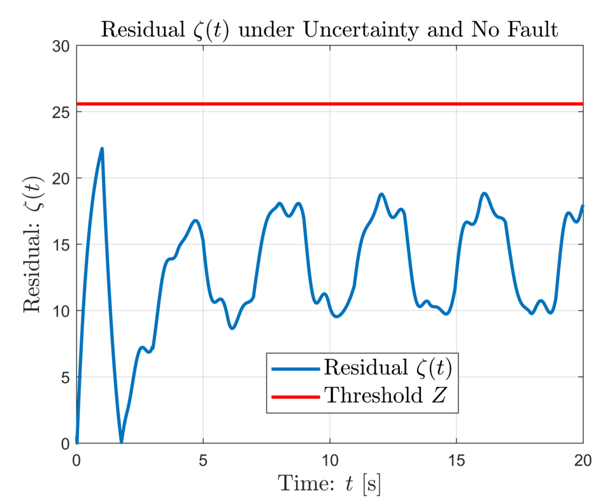

Fig. 3. Residual evolution under uncertain but fault-less case. The residual remains bounded within the threshold as there is no fault.

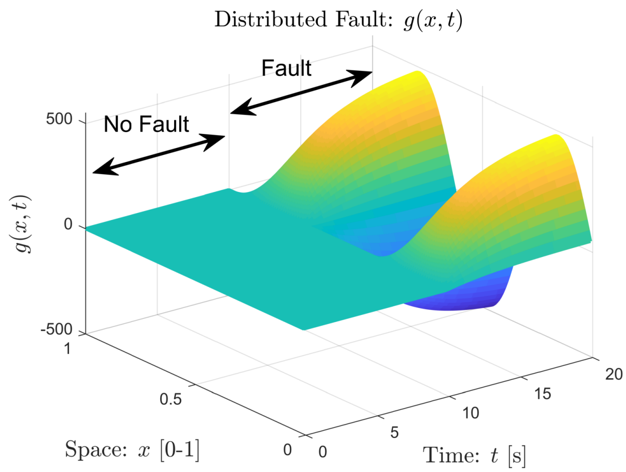

Fig. 4. Distributed fault $g(x,t)$ injected to the plant at $t = 10$ sec.

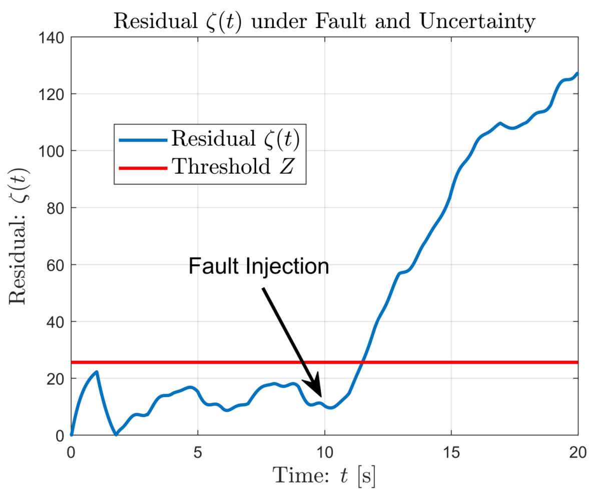

Fig. 5. Residual evolution under uncertainty and fault. The fault is injected at $t = 10$ sec. The residual indicates the fault occurrence by crossing the threshold at $t = 11.8$ sec.

V. CONCLUSIONS

In this paper, we design a robust fault diagnosis scheme for wave equations with boundary measurements. The scheme consists of a boundary output injection-based backstepping PDE observer. The output error of the PDE observer is treated as fault indicating *residual signal*. We have analyzed the convergence of the PDE observer and the *residual signal* via Lyapunov stability theory. We have proved that the *residual signal* will remain bounded in the presence of fault and/or uncertainty. Furthermore, we have derived a threshold with which the *residual signal* is compared to infer fault occurrence. Essentially, the threshold provides robustness by separating the effect of uncertainties from faults. We have performed simulation studies to illustrate the performance of the scheme.

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